Research Laboratories

UNITED AIRCRAFT CORPORATION

EAST HARTFORD, CONNECTICUT

Report C-910098-12

Optimal Variable-Thrust Rendezvous of a Power-Limited Rocket Between Neighboring Low-Eccentricity Orbits

Contract NAS8-11099

Final Report

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DATE October 1964

NO. OF PAGES 65

COPY NO. 6

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SUMMARY

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A study has been made of minimum-fuel transfer and rendezvous between neighboring low-eccentricity orbits by power-limited rocket. This study includes and extends previous work wherein only the case of transfer between circular orbits was considered. As before, the analysis is based on the assumption that only small deviations from an initial orbit are allowed. Complete analytical solutions are obtained in three different sets of variables: (1) rotating rectangular coordinates, (2) rotating spherical coordinates, and (3) Lagrange's planetary variables. In addition to the determination of optimal transfer and rendezvous trajectories in three dimensions, synthesis of the optimal controls is also carried out in each case. The guidance coefficients resulting from the control synthesis are presented both in graphical form and in equation form suitable for use in guidance applications.

The use of an intermediate reference orbit is found to be a powerful method of improving the accuracy of the linearized theory. Results for circular, coplanar earth-Venus and earth-Mars transfers are compared with exact solutions. The linear theory is shown to provide a very good correlation with exact data for all trip times of interest.

CONCLUSIONS

1. Explicit solutions are obtainable for minimum-fuel transfer and rendezvous between neighboring low-eccentricity orbits by power-limited rockets. These solutions include closed form expressions for the optimum thrust vector, the optimum trajectory, and the minimum required fuel consumption in terms of boundary conditions and trip time.

2. Synthesis of the optimal control has also been carried out for both transfer and rendezvous between any orbit and a neighboring, low-eccentricity orbit. Guidance coefficients for each case can be presented in terms of time remaining to reach the target orbit.

3. Results for the case of coplanar circle-to-circle transfer between earth and Venus indicate that the linearized equations adequately predict the actual motion, the optimal control, and the minimum fuel consumption. There is, as yet, no numerical data to indicate that the rendezvous equations are equally applicable to the planetary orbits. The failure of these equations appears to be caused by the terms representing the angular motion.

RECOMMENDATIONS

The results of the linearized analysis for earth-Mars and earth-Venus transfers are sufficiently promising to warrant further investigation into higher-order theories. In particular, the "piecewise-linear" theory described herein is a relatively straightforward application of the linearized equations which should include at least some second-order effects on the motion. It is recommended that this approach be pursued because a simple second-order solution is highly desirable.

INTRODUCTION

It is characteristic of high-specific-impulse, low-thrust propulsion systems that the source of power is separate from the thrust device itself. Consequently, such propulsion systems are referred to as power-limited, since thrust is restricted in magnitude by the output of the power supply, which is in turn limited by the necessity of minimizing power supply weight.

The problem of transfer and rendezvous between neighboring orbits by a power-limited rocket is of interest for two basic reasons. First of all, the problem can be solved analytically, as was demonstrated in Refs. 1, 2, and 3, provided that the thrust acceleration is not constrained in magnitude and that the proper simplifying assumptions are made in the mathematical model of the system. The analytic expressions thus obtained for the controls and for the optimum trajectories then provide insight into more general problems where the simplifying restrictions are lifted. Secondly, the solution to this problem provides a lower bound to the performance requirements for low-thrust orbital transfer and rendezvous.

It is interesting to note that if, for the same system model as has been used herein, the thrust acceleration is assumed constant, analytic integration of the equations of motion requires the evaluation of incomplete elliptic integrals of the third kind (Ref. 4). Therefore, allowance for variable-thrust acceleration is essential if simple analytic solutions are to be obtained.

ANALYTICAL METHOD

Description of the Mathematical Model

The phrase "neighboring orbits", as defined here, requires that the inclination between orbit planes be small and that the radial separation between orbits be small relative to the semi-major axis of either orbit. If it is further assumed that motion in the transfer orbit does not deviate significantly from these neighboring orbits, linearization of the equations of motion is permissible.

The analysis has been carried out in three sets of variables: (1) rotating rectangular coordinates, (2) rotating spherical coordinates, and (3) Lagrange's planetary variables. The rotating coordinates have been utilized previously in Refs. 5, 6, and 7, while the planetary variables were applied to an orbit transfer problem in Ref. 4.

The rotating coordinate systems are depicted in Figs. 1 and 2. Each consists of an origin which revolves at satellite velocity in the initial (interior) circular orbit and orthogonal coordinates measured from this revolving origin. In the rectangular system of Fig. 1, y' is a radial dimension, x' is measured tangent to the initial orbit at the origin, and z' is a coordinate which is out of the plane of the initial orbit and is normal to both x' and y'.

In Fig. 2, the spherical system is composed of a radial coordinate y, an arc x, measured circumferentially from the origin, and another arc z, which is orthogonal to the x-y plane.

The Lagrange planetary variables, which are derived from the elements of an elliptic orbit and are used in the standard variation-of-parameters equations of celestial mechanics (Ref. 8), are convenient because they eliminate the necessity of treating singularities for zero eccentricity and zero inclination in these equations. As they are used in this study, the planetary variables consist of the nondimensionalized semi-major axis $x_1 = a/a_0$, a circumferential distance component, x_4, and the following combinations of the remaining orbital elements:

$$\begin{aligned} x_2 &= e \sin \omega \\ x_3 &= e \cos \omega \\ x_5 &= \sin i \sin \Omega \\ x_6 &= \sin i \cos \Omega \end{aligned} \tag{1}$$

where e is eccentricity, ω is the longitude of peri-apsis, i is orbital inclination, and Ω is the longitude of the ascending node. The planetary variables provide a simple means of introducing eccentricity into the terminal orbits, and the form of the state equations using these variables is particularly simple in the present problem. However, in a practical application, they might be less desirable than the rotating coordinates because the orbital elements cannot be directly measured.

In view of the foregoing considerations, eccentric terminal orbits have been allowed only in the planetary variables in this study, while the analysis in rotating reference frames is confined to circular terminal orbits.

It should be noted here that the three sets of variables are entirely equivalent in that the equations of motion may be transformed directly from one set to another by substitution. There are some differences in the required linearizing assumptions which should be mentioned, however.

Consider the coordinate system depicted in Fig. 1, a rectangular system with its origin fixed on the interior orbit (assumed to be the reference orbit) in the x', y' plane. The mutually orthogonal coordinates x', y', and z' form a triad that revolves with angular speed n_o characteristic of the reference orbit, so that motion in this frame of reference is relative to a point on the reference orbit. The spherical coordinate system in Fig. 2 is described by the arc x in the plane of the reference orbit, the arc z measured normal to this plane, and a radial dimension y.

In order to linearize the equations of motion in the first system, it is necessary to assume that excursions x', y', and z' from the origin be small in comparison with the radius, r_o, of the reference orbit. Motion is therefore constrained to a small sphere about the origin. No restrictions are placed on the component velocities. In the rotating spherical system, only the assumption of small component velocities will linearize the equations, whereas the arc x is not limited. The resultant motion is constrained to a torus about the reference orbit.

Since the linearized equations of motion are identical except for differences in notation (Ref. 5), one can draw the conclusion that, if in the spherical system the resultant motion does not involve large variations in x, the velocity components may be large. In the present study, use of the spherical system has been assumed throughout, and the results may be extended according to the foregoing discussion.

In the case of the planetary variables, the linearizing assumptions require that the difference in the semi-major axes of the terminal orbits be small and that the eccentricity of the terminal orbits as well as the eccentricity of the instantaneous transfer orbit be small. The implications of these assumptions are similar to those for the rotating spherical system in that "fast" trajectories are allowed only when the linearizing assumptions may be relaxed. On the other hand, fast trajectories are allowed in the rectangular system because no limits are placed on the component velocities in the linearizing process.

Analysis

The optimization problem is to derive the optimal control equation for the minimum-fuel transfer or rendezvous of a power-limited rocket between neighboring orbits in a given time. Mathematically, this requires minimization of the integral

$$J = \int_0^{t_f} (T/m)^2 \, dt = \int_0^{\tau_f} (n_0/2) \, A^2 \, d\tau = \int_0^{\tau_f} f_0 \, (A) \, d\tau \qquad (2)$$

subject to constraints imposed by the equations of state which may be expressed in the form

$$\dot{x}_i = f_i \, (x, A) \qquad i = 1, \ldots, n \qquad (3)$$

The control is the thrust acceleration vector, A, in the present case.

The problem is treated as a problem of Lagrange in the calculus of variations. In particular, Breakwell's formulation (Ref. 9) of this problem is used because the linearized equations in the present case are particularly well suited to this formulation.

If a fundamental function F is defined as

$$F = -f_0 + \sum_{i=1}^{n} \lambda_i f_i \qquad (4)$$

the variational treatment requires satisfaction of Euler-Lagrange equations in the following form as necessary conditions for the existence of an extremal arc:

$$\frac{d\lambda_i}{d\tau} = - \frac{\partial F}{\partial x_i} \qquad (5)$$

$$\frac{\partial F}{\partial A_j} = 0 \qquad (6)$$

An additional necessary condition provided by the Pontryagin Maximum Principle must also be satisfied to ensure that the stationary solution predicted by the Euler equations is actually an extremum. The maximum principle, which may be expressed as

$$F\ (x_i,\ \lambda_i,\ A_j^*) \geq F\ (x_i,\ \lambda_i,\ A_j) \tag{7}$$

ensures that the stationary solution is an absolute maximum. Furthermore, it has been shown (Ref. 10) that for a system where both the state variables and the controls appear linearly in the state equations, the maximum principle is also sufficient to ensure a minimum of the payoff, J. Since all cases in the present analyses are linear in the controls and satisfy the maximum principle, the optimum trajectories described herein are absolute extrema.

Due to the great number of equations involved, the variational analysis is not described in each case. Only the most important equations are included, and these are grouped in an orderly fashion in the appendixes. The rotating coordinate systems are considered in Appendix I, and the planetary variables are considered in Appendix II. For a more detailed account of the application of the aforementioned equations the reader is referred to Ref. 2 wherein a specific case is treated in detail.

Synthesis of the Optimal Controls

In order to put the equations for the optimized controls into a form compatible with guidance requirements, several changes are made. First, τ in the control equations is replaced by $-\tau$. That is, the equations are rewritten with "time-to-go" as the independent variable. Secondly, while in the ordinary transfer and rendezvous analyses in rotating coordinates it was generally convenient to assume zero initial conditions, the terminals are reversed in the control synthesis. That is, the target orbit is assumed to be defined by zero values in most of the state variables. The results of the control synthesis are expressed in terms of the guidance coefficients, $\partial A_j/\partial x_i$, of each component of the control vector, A.

The equations for the control synthesis are summarized in Appendix III for transfer and rendezvous in each of the coordinate systems. Those equations which deal specifically with transfer between circular orbits have been presented previously in Ref. 3.

RESULTS

Orbit Transfer and Rendezvous

The multiplicity of solutions generated in this study (particularly for rendezvous) precludes a graphical presentation of all the resulting trajectories. An attempt is made to summarize the results in a reasonably concise form with orbit transfer solutions represented as special cases of rendezvous wherever feasible.

To simplify the presentation of the results, only circle-to-circle transfer and rendezvous cases are examined in the summary curves of Figs. 3 through 13. The first set of plots, Figs. 3 through 5, shows the variation of the components of the optimal thrust acceleration with time for circle-to-circle transfer only.

The in-plane components A_x/y_f and A_y/y_f are seen to display symmetry about the midpoint in time for all trip times, as does the out-of-plane component $A_z/r_0 i$. In particular, when $\tau_f = 2n\pi$, the components A_x/y_f and A_y/y_f are constant with time, and the latter is zero. For the coplanar problem, constant circumferential thrust acceleration is thereby specified as the optimum mode for integral multiples of the period of the reference orbit, a result that is in agreement with Ref. 7.

Figures 6 through 8 show the thrust acceleration components for circle-to-circle rendezvous at a particular trip time equal to one sixth of an orbital period of the reference orbit. The parameter in Figs. 6 and 7 is $x_f/y_f\tau_f$ which takes on the value of 3/4 for the special case of optimum transfer. Similarly the out-of-plane component is plotted with Ω_f as a parameter. As indicated, the longitude of the node can have either of two values, 150 or 330 deg, for optimum transfer.

The payoff, J, can be best represented as the sum of three components, J_1, J_2, and J_3, which are defined by Eqs. (A-44)and(A-45) and are plotted in Figs. 9 through 11. The components J_1 and J_2 define propellant requirements for coplanar rendezvous, while the addition of J_3 introduces the out-of-plane requirement. In particular J is equal to J_1 for coplanar transfer since the term $x_f/y_f\tau_f - 3/4$ in J_2 is zero for optimum transfer.

All three components, as well as their sum, are seen to be monotonically decreasing functions of τ_f. In the limit,as $\tau_f \to \infty$, A and J $\to$ 0. This is a consequence of the fact that no limit has been placed on exhaust velocity. Similarly all three components tend to infinity as τ_f approaches zero because zero trip time requires infinite thrust acceleration.

An interesting feature of J_3 is evident from Fig. 11. For τ_f=kπ,where k = 0, 1, 2, ..., J_3 is the same for all nodal longitudes, Ω_f. For all other times the envelope of the family of curves is given by the equations

$$J_{3_{max}} = \frac{1}{\tau_f - |\sin \tau_f|} \tag{8}$$

$$J_{3_{min}} = \frac{1}{\tau_f + |\sin \tau_f|} \tag{9}$$

where the lower envelope is given by Eq. (9) and represents J_3 for optimum transfer.

Choice of Reference Orbit

It has been observed that the linearized equations are applicable only for orbits which are not separated by large radial distances. More specifically, excursions from the origin in the y direction should always be small. It is apparent, however, that when the reference orbit is chosen to have the same radius as the initial orbit the excursion, y, to the final orbit is maximized. A better reference orbit would be one midway between the terminal orbits since this device would guarantee a radial excursion no greater than half the distance between the terminals.

Although for the most part, the equations of this report are based on a reference orbit coincident with the initial orbit, Eqs. (A-48) through (A-51) and (A-131) through (A-134) are exceptions in this respect. These equations are derived to account for an arbitrary choice of the reference orbit and may therefore be applicable in cases where the ordinary equations break down.

Application to Planetary Orbits

Strictly speaking, none of the planetary orbits are "neighboring orbits" in the sense in which this term has been defined. Earth's closest neighbor, Venus, has a semi-major axis, $a = 0.7233$AU compared with $a = 1.0$AU for earth, leaving a separation distance of 0.2767AU which is not $\ll$ 1.0AU. However, using the improvement referred to above, it is possible to apply the linearized analysis to earth-Venus and earth-Mars trajectories with remarkably good accuracy. In Figs. 12 and 13, comparisons have been made with exact solutions from Ref. 11, for earth-Venus and earth-Mars transfers. The circled points were calculated from Eq. (A-48) of Appendix I using a reference orbit midway between the two terminal orbits. These results for the special case of uninclined, circular terminal orbits show only a slight discrepancy in J for transfer times up to one earth year.

Extension of the Linearized Theory

Based on the successful correlation indicated by Figs. 12 and 13, a new theory is being considered in order to account for second-order effects in J. This theory is a "piecewise-linear" analysis which may be described as follows: The transfer (or rendezvous) is divided into two steps, each requiring a portion of the total trip time. The first segment of the trajectory consists of a rendezvous from the initial orbit to an intermediate orbit of unspecified size and shape, and the second segment is a rendezvous from this intermediate orbit to the final terminal orbit. The expression for J is composed of two linear expressions for the two segments, and the parameters of the intermediate orbit are considered as variables which may be optimized so as to minimize the total J. In each segment an appropriate reference orbit is chosen so as to improve the accuracy of the theory.

This approach should provide better results than the linearized theory. Since the results for earth-Mars and earth-Venus transfers were already good, the piecewise-linear theory may approach exact results in these cases and might even yield acceptable results for trajectories to the outer planets.

Control Synthesis

In this study it has been possible to express each of the components of the optimal control vector, A, as a linear function of the n state variables.

$$A_j = \sum_{i=1}^{n} \frac{\partial A_j}{\partial x_i} x_i \tag{10}$$

Therefore, the presentation of the results can be confined to curves of the guidance coefficients, $\partial A_j/\partial x_i$ plotted against time to go, τ'. Using the equations for the guidance coefficients which comprise Appendix III, the summary curves of Figs. 14 through 25 were generated.

The synthesized controls for the case of transfer between an arbitrary state and a nearby circular orbit appear in Figs. 14 through 16 in terms of the rotating coordinate system variables. The extension to include eccentricity of the final orbit is provided by use of the Lagrange planetary variables in Figs. 17 through 19.

For rendezvous the same procedure is followed in the presentation of the synthesized controls, with the addition of curves to account for the dependence of in-plane thrust acceleration components on the circumferential distance. In rotating coordinates, Figs. 20 through 22 summarize the results for rendezvous between any initial state and a point on a nearby circular orbit.

As in the transfer case, the planetary variables facilitate the extension to rendezvous between an initial state and a point on a nearby orbit of low eccentricity. The results for the planetary variables appear in Figs. 23 through 25.

All the curves for the guidance coefficients display similar behavior. When time-to-go is short, the curves diverge to infinity (either positive or negative), but a damped oscillation is evident, causing the coefficients to approach zero for very long times.

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LIST OF SYMBOLS

$\frac{T}{m}$	Thrust-to-mass ratio
A	$\frac{1}{n_o}\frac{T}{m}$
C	Integration constant
f	Rate of change of a state variable
F	Fundamental function
J	Defined by Eq. (2)
D	Defined by Eq. (A-154)
B	Defined by Eq. (A-182)
Q	Defined by Eq. (A-181)
Φ	Defined by Eq. (A-146)
λ	Lagrange multiplier
r	Radius
R	Radial force
W	Normal force
S	Circumferential force
n	Mean angular motion
x, y, z	Position components in spherical system
x', y', z'	Position components in rectangular system
u, v, w	Velocity components in x, y, z, directions
t	Time
τ	$n_o t$

LIST OF SYMBOLS
(contd.)

τ'	Time to go
η	True anomaly
ω	Longitude of peri-apsis
e	Eccentricity
N	Unit vector normal to instantaneous transfer orbit
a	Semi-major axis
Ω	Longitude of the node
i	Inclination
x_1	a/a_o
x_2	$e \sin \omega$
x_3	$e \cos \omega$
x_5	$\sin i \sin \Omega$
x_6	$\sin i \cos \Omega$
$\vec{c}$	Angular momentum vector

Subscripts

i	Index denoting x, y, z, u, v, w
j	Index denoting x, y, z
o	Initial condition
f	Final condition
x,y,z,u,v,w	Denoting state variable
I	Intermediate reference orbit
R	Radial

LIST OF SYMBOLS
(contd.)

S	Circumferential
W	Normal

Superscripts

*	Optimum condition
→	Denotes a vector

APPENDIX I

ROTATING RECTANGULAR AND SPHERICAL COORDINATE SYSTEMS

1. Equations of State

$$\frac{dx}{d\tau} = u \quad \text{(A-1)}$$

$$\frac{dy}{d\tau} = v \quad \text{(A-2)}$$

$$\frac{dz}{d\tau} = w \quad \text{(A-3)}$$

$$\frac{du}{d\tau} = A_x + 2y \quad \text{(A-4)}$$

$$\frac{dv}{d\tau} = A_y + 3y - 2u \quad \text{(A-5)}$$

$$\frac{dw}{d\tau} = A_z - z \quad \text{(A-6)}$$

2. Euler-Lagrange Equations

$$\dot{\lambda}_x = 0 \quad \text{(A-7)}$$

$$\dot{\lambda}_y = -3\lambda_v \quad \text{(A-8)}$$

$$\dot{\lambda}_z = \lambda_w \quad \text{(A-9)}$$

$$\dot{\lambda}_u = -\lambda_x + 2\lambda_v \quad \text{(A-10)}$$

$$\dot{\lambda}_v = -\lambda_y - 2\lambda_u \quad \text{(A-11)}$$

$$\dot{\lambda}_w = -\lambda_z \quad \text{(A-12)}$$

$$\lambda_u = n_0 A_x \quad \text{(A-13)}$$

$$\lambda_v = n_0 A_y \quad \text{(A-14)}$$

$$\lambda_w = n_0 A_z \quad \text{(A-15)}$$

3. Integrated Euler-Lagrange Equations

$$\lambda_x = n_0 C_0 \tag{A-16}$$

$$\lambda_y = -6n_0(C_4 + C_0\tau - C_1\cos\tau + C_2\sin\tau) \tag{A-17}$$

$$\lambda_z = 2n_0(C_5\sin\tau + C_3\cos\tau) \tag{A-18}$$

$$\lambda_u = n_0(3C_4 + 3C_0\tau - 4C_1\cos\tau + 4C_2\sin\tau) \tag{A-19}$$

$$\lambda_v = 2n_0(C_0 + C_1\sin\tau + C_2\cos\tau) \tag{A-20}$$

$$\lambda_w = 2n_0(C_5\cos\tau - C_3\sin\tau) \tag{A-21}$$

4. Boundary Conditions

	Transfer		Rendezvous	
State Variable	$\tau = 0$	$\tau = \tau_f$	$\tau = 0$	$\tau = \tau_f$
x	0	FREE	0	x_f
y	0	y_f	0	y_f
z	0	z_f	0	z_f
u	0	$\frac{3}{2}y_f$ (1)	0	$\frac{3}{2}y_f$ (1)
v	0	0	0	0
w	0	$\sqrt{r_0^2 i^2 - z_f^2}$ (2)	0	$\sqrt{r_0^2 i^2 - z_f^2}$ (2)

5. Integrated Equations of State (with initial conditions)

$$\begin{aligned} x = {} & \left[16(\tau - \sin\tau) - \frac{3}{2}\tau^3\right]C_0 + \left[16(1-\cos\tau) - 10\tau\sin\tau\right]C_1 \\ & + \left[22\sin\tau - 10\tau\cos\tau - 12\tau\right]C_2 - \left[\frac{9}{2}\tau^2 - 12(1-\cos\tau)\right]C_4 \end{aligned} \tag{A-22}$$

$$\begin{aligned} y = {} & \left[8(1-\cos\tau) - 3\tau^2\right]C_0 + 5\left[\sin\tau - \tau\cos\tau\right]C_1 + \left[5\tau\sin\tau - 8(1-\cos\tau)\right]C_2 \\ & + 6\left[\sin\tau - \tau\right]C_4 \end{aligned} \tag{A-23}$$

(1) REF 6

(2) REF 5

$$z = \left[\tau \cos\tau - \sin\tau\right] C_3 + \left[\tau \sin\tau\right] C_5 \tag{A-24}$$

$$u = \left[16(1-\cos\tau) - \frac{9}{2}\tau^2\right] C_0 + \left[6\sin\tau - 10\tau\cos\tau\right] C_1 + \left[10\tau\sin\tau - 12(1-\cos\tau)\right] C_2 + \left[12\sin\tau - 9\tau\right] C_4 \tag{A-25}$$

$$v = \left[8\sin\tau - 6\tau\right] C_0 + \left[5\tau\sin\tau\right] C_1 + \left[5\tau\cos\tau - 3\sin\tau\right] C_2 + 3\left[1-\cos\tau\right] C_4 \tag{A-26}$$

$$w = \left[-\tau\sin\tau\right] C_3 + \left[\sin\tau + \tau\cos\tau\right] C_5 \tag{A-27}$$

6. Transversality Conditions - Transfer

$$\lambda_x = C_0 = 0 \tag{A-28}$$

$$\frac{C_5}{C_3} = \frac{\tan\tau_f + \frac{w_f}{z_f}}{1 - \frac{w_f}{z_f}\tan\tau_f} \tag{A-29}$$

7. Constants of Integration - Transfer

$$C_1 = \frac{y_f \sin\tau_f}{16(1-\cos\tau_f) - \tau_f(5\tau_f + 3\sin\tau_f)} \tag{A-30}$$

$$C_2 = \frac{-y_f(1-\cos\tau_f)}{16(1-\cos\tau_f) - \tau_f(5\tau_f + 3\sin\tau_f)} \tag{A-31}$$

$$C_3 = \frac{(\sin\tau_f + \tau_f\cos\tau_f)z_f - (\tau_f\sin\tau_f)\sqrt{r_o^2 i^2 - z_f^2}}{\tau_f^2 - \sin^2\tau_f} \tag{A-32}$$

$$C_4 = \frac{\frac{y_f}{6}(5\tau_f + 3\sin\tau_f)}{16(1-\cos\tau_f) - \tau_f(5\tau_f + 3\sin\tau_f)} \tag{A-33}$$

Rendezvous

$$C_0 = \frac{\tau_f\, y_f\left(\frac{x_f}{y_f\tau_f} - \frac{3}{4}\right)(5\tau_f - 3\sin\tau_f)}{\frac{3}{4}\tau_f(5\tau_f - 3\sin\tau_f)(\tau_f^2 - 80) + 4(1-\cos\tau_f)(71\tau_f^2 - 64) + 248\tau_f^{\ 2}\cos\tau_f} \tag{A-34}$$

$$C_1 = \frac{y_f\sin\tau_f}{16(1-\cos\tau_f) - \tau_f(5\tau_f + 3\sin\tau_f)} + C_0\left[\frac{3\sin\tau_f - 8(1-\cos\tau_f)}{5\tau_f - 3\sin\tau_f}\right] \tag{A-35}$$

$$C_2 = \frac{-y_f(1-\cos\tau_f)}{16(1-\cos\tau_f) - \tau_f(5\tau_f + 3\sin\tau_f)} + C_0\left[\frac{3\tau_f(1+\cos\tau_f) - 8\sin\tau_f}{5\tau_f - 3\sin\tau_f}\right] \tag{A-36}$$

$$C_3 = \frac{(\sin\tau_f + \tau_f\cos\tau_f)\,z_f - (\tau_f\sin\tau_f)\sqrt{r_0^{\ 2}\,i^2 - z_f^{\ 2}}}{(\tau_f^{\ 2} - \sin^2\tau_f)} \tag{A-37}$$

$$C_4 = \frac{\frac{y_f}{6}(5\tau_f + 3\sin\tau_f)}{16(1-\cos\tau_f) - \tau_f(5\tau_f + 3\sin\tau_f)} - C_0\frac{\tau_f}{2} \tag{A-38}$$

$$C_5 = \frac{(\tau_f\sin\tau_f)\,z_f + (\tau_f\cos\tau_f - \sin\tau_f)\sqrt{r_0^{\ 2}\,i^2 - z_f^{\ 2}}}{(\tau_f^2 - \sin^2\tau_f)} \tag{A-39}$$

8. Controls

$$A_x = 3C_4 + 3C_0\tau - 4C_1\cos\tau + 4C_2\sin\tau \tag{A-40}$$

$$A_y = 2\left[C_0 + C_1 \sin\tau + C_2 \cos\tau\right] \tag{A-41}$$

$$A_z = 2\left[C_5 \cos\tau - C_3 \sin\tau\right] \tag{A-42}$$

9. Payoff

Transfer

$$\frac{J}{n_o^3 r_o^2} = \frac{\left(\frac{y_f}{r_o}\right)^2 (5\tau_f + 3\sin\tau_f)}{8\left[\tau_f(5\tau_f + 3\sin\tau_f) - 16(1-\cos\tau_f)\right]} + \frac{i^2}{\tau_f + |\sin\tau_f|} \tag{A-43}$$

Rendezvous

$$\frac{J}{n_o^3 r_o^2} = J_1\left(\frac{y_f}{r_o}\right)^2 + J_2\left(\frac{y_f}{r_o}\right)^2\left(\frac{x_f}{y_f\tau_f} - \frac{3}{4}\right)^2 + J_3 i^2 \tag{A-44}$$

$$\begin{aligned}\frac{J}{n_o^3 r_o^2} = {} & \frac{\left(\frac{y_f}{r_o}\right)^2 (5\tau_f + 3\sin\tau_f)}{8\left[\tau_f(5\tau_f + 3\sin\tau_f) - 16(1-\cos\tau_f)\right]} \\ & + \frac{\frac{\tau_f^2}{2}\left(\frac{y_f}{r_o}\right)^2\left(\frac{x_f}{y_f\tau_f} - \frac{3}{4}\right)^2 (5\tau_f - 3\sin\tau_f)}{\frac{3}{4}\tau_f(5\tau_f - 3\sin\tau_f)(\tau_f^2 - 80) + 4(1-\cos\tau_f)(71\tau_f^2 - 64) + 248\tau_f^2\cos\tau_f} \\ & + i^2\left[\frac{\tau_f - \sin\tau_f\cos(2\Omega_f + \tau_f)}{(\tau_f^2 - \sin^2\tau_f)}\right]\end{aligned} \tag{A-45}$$

10. It should be pointed out that for each free end condition in the case of orbit transfer, the variational analysis predicts an optimum value for that particular state variable at the end point. In the rotating coordinate systems the x and z coordinates are left open at final time, τ_f. The end point for the optimal transfer is then determined in the analysis and is defined by the equations.

$$\left(\frac{z_f}{r_o}\right)^* = i\sqrt{\frac{1 \mp \cos\tau_f}{2}} \tag{A-46}$$

$$\left(\frac{x_f}{y_f}\right)^* = \frac{3}{4}\tau_f \tag{A-47}$$

11. Payoff Equations with an Intermediate Reference Orbit

Let the origin revolve in a circular orbit of radius r_I between the two terminal orbits such that the radial distance to the outer orbit is $r_f - r_I$ and the radial distance to the inner orbit is $r_I - r_0$. The radii r_0 and r_f refer to the inner and outer orbits, respectively.

Transfer

$$\frac{J}{n_I^3 r_I^2} = \frac{\frac{1}{8}\left(\frac{r_f - r_o}{r_I}\right)^2 (5\tau_f + 3\sin\tau_f)}{\tau_f(5\tau_f + 3\sin\tau_f) - 16(1-\cos\tau_f)} + \frac{i^2}{\tau_f + |\sin\tau_f|} \tag{A-48}$$

Rendezvous

$$\begin{aligned}\frac{J}{n_I^3 r_I^2} = {} & \frac{\frac{1}{8}\left(\frac{r_f - r_o}{r_I}\right)^2 (5\tau_f + 3\sin\tau_f)}{\tau_f(5\tau_f + 3\sin\tau_f) - 16(1-\cos\tau_f)} \\ & + \frac{\frac{\tau_f^2}{2}\left\{\frac{x_f}{\tau_f r_I} - \frac{3}{4}\left(\frac{r_f + r_o}{r_I} - 2\right)\right\}^2 (5\tau_f - 3\sin\tau_f)}{\frac{3}{4}\tau_f(5\tau_f - 3\sin\tau_f)(\tau_f^2 - 80) + 4(1-\cos\tau_f)(71\tau_f^2 - 64) + 248\tau_f^2\cos\tau_f} \\ & + i^2\left\{\frac{\tau_f^2 - \sin\tau_f\cos(2\Omega_f + \tau_f)}{\tau_f^2 - \sin^2\tau_f}\right\}\end{aligned} \tag{A-49}$$

12. Optimal Transfer Coordinates

$$\left(\frac{x_f}{r_I}\right)^* = \frac{3}{4}\tau_f\left(\frac{r_f + r_o}{r_I} - 2\right) \tag{A-50}$$

$$\left(\frac{z_f}{r_I}\right)^* = i\sqrt{\frac{1 \pm \cos\tau_f}{2}} \tag{A-51}$$

APPENDIX II

LAGRANGE'S VARIABLES

In the theory of special perturbations, as derived in Ref. 8 for example, the equations for rates of change of the elements of an elliptic orbit are written in terms of the elements and acceleration components S, R, and W, which are perpendicular to the radius vector, radial and normal to the orbital plane, respectively.

Consider the five elements, a, e, i, ω, Ω. The equations for small rates of change of these variables are

$$\frac{da}{dt} = \frac{2}{n\sqrt{1-e^2}}\left[eR\sin\eta + S(1+e\cos\eta)\right] \tag{A-52}$$

$$\frac{de}{dt} = \frac{\sqrt{1-e^2}}{na}\left[R\sin\eta + \frac{2\cos\eta + e + e\cos^2\eta}{1+e\cos\eta}\,S\right] \tag{A-53}$$

$$\frac{di}{dt} = \frac{\sqrt{1-e^2}}{na}\,W\cos(\omega+\eta) \tag{A-54}$$

$$\frac{d\omega}{dt} = \frac{\sqrt{1-e^2}}{nae}\left[-R\cos\eta + \frac{2+e\cos\eta}{1+e\cos\eta}\,S\sin\eta - \frac{e\tan\frac{i}{2}\sin(\omega+\eta)}{1+e\cos\eta}\,W\right] \tag{A-55}$$

$$\frac{d\Omega}{dt} = \frac{\sqrt{1-e^2}}{na}\,\frac{W}{\sin i}\,\sin(\omega+\eta) \tag{A-56}$$

In order to avoid singularities for zero eccentricity and inclination in Eqs. (A-55) and (A-56) these equations may be transformed according to the following definitions:

$$x_2 = e\sin\omega \tag{A-57}$$

$$x_3 = e \cos\omega \tag{A-58}$$

$$x_5 = \sin i \sin\Omega \tag{A-59}$$

$$x_6 = \sin i \cos\Omega \tag{A-60}$$

Under the assumptions

$$\begin{aligned} e &<< 1 \\ a &\approx a_o \\ n &\approx n_o \\ \tau = n_o t &= \omega + \eta \\ i &<< 1 \end{aligned} \tag{A-61}$$

$$A_R = \frac{R}{a_o n_o^2}, \quad A_S = \frac{S}{a_o n_o^2}, \quad A_W = \frac{W}{a_o n_o^2} \tag{A-62}$$

and with the further definitions

$$x_1 = \frac{a}{a_o} \tag{A-63}$$

$$x_4 = x \tag{A-64}$$

the equations of state for the variational problem may be derived from Eqs. (A-52) through (A-60).

There is a direct equivalence between these equations and the equations of state in the rotating coordinate system variables. That is, each of the Lagrange variables x_1, x_2, x_3, ..., x_6, can be expressed in terms of the rotating coordinate variables, x, y, z, u, v, and w.

Referring to Fig. 26, define a position vector $\vec{r}$ in nonrotating coordinates originating at the center of attraction F. Assume the motion out of the reference plane is uncoupled from the in-plane motion.

Relative to a rotating rectangular coordinate system originating at O and rotating with angular velocity $\vec{n}$ this vector is

$$\vec{r} = x\vec{i} + (r_0 + y)\vec{j} \tag{A-65}$$

where the unit vectors $\vec{i}$ and $\vec{j}$ are taken in the x and y directions, respectively. The vector velocity $\vec{V}$ is obtained by differentiating $\vec{r}$.

$$\vec{V} = \frac{d\vec{r}}{dt} = u\vec{i} + v\vec{j} + \vec{n} \times \vec{r} \tag{A-66}$$

Since $\vec{n} = n_o\vec{k}$, the expression for $\vec{V}$ is

$$\vec{V} = \left[u - n_o(r_o + y)\right]\vec{i} + (v + n_o x)\vec{j} \tag{A-67}$$

Using Eqs. (A-65) and (A-67), expressions can be written for the angular momentum $\vec{C}$, the path speed V and the radius r of the vehicle

$$\vec{C} = \vec{r} \times \vec{V} = \left[x(v + n_o x) - (r_o + y)(u - n_o(r_o + y))\right]\vec{k} \tag{A-68}$$

$$V = \sqrt{\vec{V} \cdot \vec{V}} = \sqrt{\left[u - n_o(r_o + y)\right]^2 + \left[v + n_o x\right]^2} \tag{A-69}$$

$$r = \sqrt{\vec{r} \cdot \vec{r}} = \sqrt{x^2 + (r_o + y)^2} \tag{A-70}$$

The following equations can be written for the angular momentum, speed, and radius of a body in an inverse square field.

$$|\vec{C}| = \sqrt{Ka(1 - e^2)} \tag{A-71}$$

$$V = \sqrt{K\left(\frac{2}{r} - \frac{1}{a}\right)} = \sqrt{(\dot{r})^2 + (r\dot{\eta})^2} \tag{A-72}$$

$$r = \frac{a(1 - e^2)}{1 + e\cos\eta} \tag{A-73}$$

Combining these equations with the absolute value of $\vec{C}$, and with V and r from Eqs. (A-68), (A-69), and (A-70), the following scalar equations result.

$$\frac{a}{a_0} = \left(1 + \frac{y}{r_0}\right)\left(1 + e\cos\eta\right) \tag{A-74}$$

$$\frac{u}{n_0 r_0} - \left(1 + \frac{y}{r_0}\right) = \frac{\sqrt{\frac{a}{a_0}}}{1 + \frac{y}{r_0}} \tag{A-75}$$

$$\frac{v}{n_0 r_0} + \frac{x}{r_0} = \sqrt{\frac{e\cos\eta}{\frac{a}{a_0}}} \tag{A-76}$$

Finally, noting that

$$\begin{gathered} \frac{a}{a_0} = x_1 \ , \quad x_2 = e\sin\omega \ , \quad x_3 = e\cos\omega \\ e\cos\eta = e\cos(\tau - \omega) = x_2\sin\tau + x_3\cos\tau \end{gathered} \tag{A-77}$$

the equations relating the coordinates are obtained.

$$\frac{y}{r_0} = (x_1 - 1) - x_2\sin\tau - x_3\cos\tau \tag{A-78}$$

$$\frac{v}{n_0 r_0} = x_3\cos\tau - x_2\sin\tau \tag{A-79}$$

$$\frac{u}{n_0 r_0} = \frac{3}{2}(x_1 - 1) - 2x_2\sin\tau - 2x_3\cos\tau \tag{A-80}$$

The components of the out-of-plane motion can be related in the following way. If $\vec{N}$ is a unit vector normal to the instantaneous transfer orbit and $\vec{s}$ is a unit vector in the direction of the line of nodes, then

$$\vec{s} = \vec{N} \times \vec{k} \tag{A-81}$$

and, since the angle between $\vec{s}$ and the vehicle is $\tau - \Omega$,

$$\cos(\tau - \Omega) = \vec{s} \cdot \vec{i} \tag{A-82}$$

Also, the orbital inclination is

$$\cos i = \vec{N} \cdot \vec{k} \tag{A-83}$$

Using these parameters the equation for the elevation, z, of the probe is

$$\frac{z}{r_o} = \tan i \sin(\tau - \Omega) \simeq \sin i \sin(\tau - \Omega) \tag{A-84}$$

or

$$\frac{z}{r_o} = - x_5 \cos\tau + x_6 \sin\tau \tag{A-85}$$

The out-of-plane velocity, w, is

$$\frac{w}{n_o r_o} = x_5 \sin\tau + x_6 \cos\tau \tag{A-86}$$

1. Equations of State

$$\frac{dx_1}{d\tau} = 2A_S \tag{A-87}$$

$$\frac{dx_2}{d\tau} = 2A_S \sin\tau - A_R \cos\tau \tag{A-88}$$

$$\frac{dx_3}{d\tau} = 2A_S \cos\tau + A_R \sin\tau \tag{A-89}$$

$$\frac{dx_4}{d\tau} = \frac{3}{2}(x_1 - 1) - 2x_2 \sin\tau - 2x_3 \cos\tau \tag{A-90}$$

$$\frac{dx_5}{d\tau} = - A_W \sin\tau \tag{A-91}$$

$$\frac{dx_6}{d\tau} = A_W \cos\tau \tag{A-92}$$

2. Euler-Lagrange Equations

$$\dot{\lambda}_1 = -\frac{3}{2}\lambda_4 \quad \text{(A-93)}$$

$$\dot{\lambda}_2 = 2\lambda_4 \sin\tau \quad \text{(A-94)}$$

$$\dot{\lambda}_3 = 2\lambda_4 \cos\tau \quad \text{(A-95)}$$

$$\dot{\lambda}_4 = \dot{\lambda}_5 = \dot{\lambda}_6 = 0 \quad \text{(A-96)}$$

$$n_0 A_S = 2(\lambda_1 + \lambda_2 \sin\tau + \lambda_3 \cos\tau) \quad \text{(A-97)}$$

$$n_0 A_R = -\lambda_2 \cos\tau + \lambda_3 \sin\tau \quad \text{(A-98)}$$

$$n_0 A_W = -\lambda_5 \sin\tau + \lambda_6 \cos\tau \quad \text{(A-99)}$$

3. Integrated Euler-Lagrange Equations

$$\lambda_1 = \lambda_{10} - \frac{3}{2}\lambda_4 \tau \quad \text{(A-100)}$$

$$\lambda_2 = \lambda_{20} - 2\lambda_4 \cos\tau \quad \text{(A-101)}$$

$$\lambda_3 = \lambda_{30} + 2\lambda_4 \sin\tau \quad \text{(A-102)}$$

$$\lambda_4 = \text{CONSTANT} \quad \text{(A-103)}$$

$$\lambda_5 = \text{"} \quad \text{(A-104)}$$

$$\lambda_6 = \text{"} \quad \text{(A-105)}$$

4. Boundary Conditions

A great simplification in the complexity of the equations can be achieved by taking advantage of the symmetry afforded by the Lagrange variables x_2 and

x_3. Therefore, in performing the integrations it will be convenient to use limits $-\tau_f/2$ to $\tau_f/2$ for the "in-plane" state variables.

State Variable ("in-plane")	Transfer $\tau = -\frac{\tau_f}{2}$	Transfer $\tau = \frac{\tau_f}{2}$	Rendezvous $\tau = -\frac{\tau_f}{2}$	Rendezvous $\tau = \frac{\tau_f}{2}$
x_1	1	$\Delta x_{1_f} + 1$	1	$\Delta x_{1_f} + 1$
x_2	x_{20}	$x_{20} + \Delta x_{2_f}$	x_{20}	$x_{20} + \Delta x_{2_f}$
x_3	x_{30}	$x_{30} + \Delta x_{3_f}$	x_{30}	$x_{30} + \Delta x_{3_f}$
x_4	x_{40}	FREE	x_{40}	$x_{40} + \Delta x_{4_f}$

(out-of-plane)	$\tau = 0$	$\tau = \tau_f$	$\tau = 0$	$\tau = \tau_f$
x_5	0	x_{5_f}	0	x_{5_f}
x_6	0	x_{6_f}	0	x_{6_f}

5. Integrated Equations of State (with initial conditions)

$$\Delta x_1 = 4\lambda_{10}(\tau + \frac{\tau_f}{2}) - 4\lambda_{20}(\cos\tau - \cos\frac{\tau_f}{2}) + 4\lambda_{30}(\sin\tau + \sin\frac{\tau_f}{2}) - 3\lambda_4(\tau^2 - \frac{\tau_f^2}{4}) \qquad \text{(A-106)}$$

$$\Delta x_2 = -4\lambda_{10}(\cos\tau - \cos\frac{\tau_f}{2}) + \frac{\lambda_{20}}{2}\left[5(\tau + \frac{\tau_f}{2}) - 3(\sin\tau\cos\tau + \frac{\sin\tau_f}{2})\right]$$
$$+ \frac{3}{2}\lambda_{30}(\sin^2\tau - \sin^2\frac{\tau_f}{2}) - 2\lambda_4\left[4(\sin\tau + \sin\frac{\tau_f}{2}) - 3(\tau\cos\tau + \frac{\tau_f}{2}\cos\frac{\tau_f}{2})\right] \qquad \text{(A-107)}$$

$$\Delta x_3 = 4\lambda_{10}(\sin\tau + \sin\frac{\tau_f}{2}) + \frac{3}{2}\lambda_{20}(\sin^2\tau - \sin^2\frac{\tau_f}{2}) \qquad \text{(A-108)}$$
$$+ \frac{\lambda_{30}}{2}\left[5(\tau + \frac{\tau_f}{2}) + 3(\sin\tau\cos\tau + \frac{\sin\tau_f}{2})\right]$$
$$- 2\lambda_4\left[4(\cos\tau - \frac{\cos\tau_f}{2}) + 3(\tau\cos\tau + \frac{\tau_f}{2}\cos\frac{\tau_f}{2})\right]$$

$$\begin{aligned}\Delta x_4 = \lambda_{10}&\left\{ 3\left(\tau + \frac{\tau_f}{2}\right)^2 - 8\left[1 - \cos\left(\tau + \frac{\tau_f}{2}\right)\right]\right\}\\
&+\lambda_{20}\left\{\left(\tau + \frac{\tau_f}{2}\right)\left[5\cos\tau + 6\cos\frac{\tau_f}{2}\right] - \frac{3}{2}\sin\left(\tau + \frac{\tau_f}{2}\right) - \frac{19}{2}\sin\tau - 8\sin\frac{\tau_f}{2}\right\}\\
&+\lambda_{30}\left\{\left(\tau + \frac{\tau_f}{2}\right)\left[6\sin\frac{\tau_f}{2} - 5\sin\tau\right] + \frac{3}{2}\cos\left(\tau + \frac{\tau_f}{2}\right) - \frac{19}{2}\cos\tau + 8\cos\frac{\tau_f}{2}\right\}\\
&+\lambda_4\left\{16\left(\tau + \frac{\tau_f}{2}\right) - 6\tau_f\left[1 - \cos\left(\tau + \frac{\tau_f}{2}\right)\right] + 3\left(\frac{\tau_f}{2}\right)^3 + \frac{9}{2}\tau\left(\frac{\tau_f}{2}\right)^2 - \frac{3}{2}\tau^3\right\}\\
&+ 2x_{20}\left[\cos\tau - \cos\frac{\tau_f}{2}\right] - 2x_{30}\left[\sin\tau + \sin\frac{\tau_f}{2}\right]\end{aligned} \qquad \text{(A-109)}$$

$$x_5 = \frac{\lambda_5}{2}\left(\tau - \sin\tau\cos\tau\right) - \frac{\lambda_6}{2}\sin^2\tau \qquad \text{(A-110)}$$

$$x_6 = -\frac{\lambda_5}{2}\sin^2\tau + \frac{\lambda_6}{2}\left(\tau + \sin\tau\cos\tau\right) \qquad \text{(A-111)}$$

6. Transversality Conditions - Transfer

$$\lambda_4 = 0 \qquad \text{(A-112)}$$

$$\frac{\lambda_5}{\lambda_6} = \tan\tau \qquad \text{(A-113)}$$

7. Constants of Integration

Transfer

$$\lambda_{10} = \frac{\frac{\Delta x_{1f}}{4}\left(5\tau_f + 3\sin\tau_f\right) - 4\Delta x_{3f}\sin\frac{\tau_f}{2}}{\tau_f\left(5\tau_f + 3\sin\tau_f\right) - 16\left(1 - \cos\tau_f\right)} \qquad \text{(A-114)}$$

$$\lambda_{20} = \frac{2\Delta x_{2f}}{5\tau_f - 3\sin\tau_f} \qquad \text{(A-115)}$$

$$\lambda_{30} = \frac{2\left[\tau_f\Delta x_{3f} - 2\Delta x_{1f}\sin\frac{\tau_f}{2}\right]}{\tau_f\left(5\tau_f + 3\sin\tau_f\right) - 16\left(1 - \cos\tau_f\right)} \qquad \text{(A-116)}$$

$$\lambda_5 = \frac{x_{5f}(\tau_f + \sin\tau_f \cos\tau) + x_{6f}\sin^2\tau_f}{2(\tau_f^2 - \sin^2\tau_f)} \tag{A-117}$$

<u>Rendezvous</u>

$$\lambda_{10} = \frac{\frac{\Delta x_{1f}}{4}(5\tau_f + 3\sin\tau_f) - 4\Delta x_{3f}\sin\frac{\tau_f}{2}}{\tau_f(5\tau_f + 3\sin\tau_f) - 16(1-\cos\tau_f)} \tag{A-118}$$

$$\begin{aligned}\lambda_{20} = {} & \frac{1}{\tau_f(5\tau_f - 3\sin\tau_f)(\frac{3}{16}\tau_f^2 + 1) - 2(8\sin\frac{\tau_f}{2} - 3\tau_f\cos\frac{\tau_f}{2})^2}\Bigg[\frac{3}{4}\tau_f\Delta x_{1f}(3\tau_f\cos\frac{\tau_f}{2} - 8\sin\frac{\tau_f}{2}) \\ & + \Delta x_{2f}\left[\frac{3}{8}\tau_f^3 + 8\tau_f - 3\tau_f(1-\cos\tau_f) - 8\sin\tau_f\right] \\ & - \left[3\tau_f\cos\frac{\tau_f}{2} - 8\sin\frac{\tau_f}{2}\right]\left[2\Delta x_{3f}\sin\frac{\tau_f}{2} + \Delta x_{4f} + 4x_{30}\sin\frac{\tau_f}{2}\right]\Bigg]\end{aligned} \tag{A-119}$$

$$\lambda_{30} = \frac{2\left[\tau_f\Delta x_{3f} - 2\Delta x_{1f}\sin\frac{\tau_f}{2}\right]}{\tau_f(5\tau_f + 3\sin\tau_f) - 16(1-\cos\tau_f)} \tag{A-120}$$

$$\begin{aligned}\lambda_4 = {} & \frac{1}{\tau_f(5\tau_f - 3\sin\tau_f)(\frac{3}{16}\tau_f^2 + 1) - 2(8\sin\frac{\tau_f}{2} - 3\tau_f\cos\frac{\tau_f}{2})^2}\Bigg[-\frac{3}{16}\tau_f\Delta x_{1f}(5\tau_f - 3\sin\tau_f) \\ & - \frac{\Delta x_{2f}}{2}\left[11\tau_f\cos\frac{\tau_f}{2} + 3\sin\frac{\tau_f}{2}(1-\cos\tau_f) - 22\sin\frac{\tau_f}{2}\right] \\ & + (5\tau_f - 3\sin\tau_f)\left[\frac{\Delta x_{3f}}{2}\sin\frac{\tau_f}{2} + \frac{\Delta x_{4f}}{4} + x_{30}\sin\frac{\tau_f}{2}\right]\Bigg]\end{aligned} \tag{A-121}$$

$$\lambda_5 = \frac{2\left\{x_{5f}(\tau_f + \sin\tau_f\cos\tau_f) + x_{6f}\sin^2\tau_f\right\}}{\tau_f^2 - \sin^2\tau_f} = \frac{2i\left[\tau_f\sin\Omega_f + \sin\tau_f\sin(\Omega_f + \tau_f)\right]}{\tau_f^2 - \sin^2\tau_f} \tag{A-122}$$

$$\lambda_6 = \frac{2\left\{x_{5f}\sin^2\tau_f + x_{6f}(\tau_f - \sin\tau_f\cos\tau_f)\right\}}{\tau_f^2 - \sin^2\tau_f} = \frac{2i\left[\tau_f\cos\Omega_f - \sin\tau_f\cos(\Omega_f + \tau_f)\right]}{\tau_f^2 - \sin^2\tau_f} \tag{A-123}$$

8. Controls

$$n_o A_S = 2\lambda_{10} - 3\lambda_4\tau + 2\lambda_{20}\sin\tau + 2\lambda_{30}\cos\tau \tag{A-124}$$

$$n_o A_R = 2\lambda_4 - \lambda_{20}\cos\tau + \lambda_{30}\sin\tau \tag{A-125}$$

$$n_o A_W = -\lambda_5 \sin\tau + \lambda_6\cos\tau \tag{A-126}$$

9. Payoff

Transfer

NOTE: (1)

$$\frac{J}{n_o^3 r_o^2} = \frac{\frac{\Delta x_{1f}^2}{8}(5\tau_f + 3\sin\tau_f) - 4\,\Delta x_{1f}\,\Delta x_{3f}\sin\frac{\tau_f}{2} + \tau_f\,\Delta x_{3f}^2}{\tau_f(5\tau_f + 3\sin\tau_f) - 16(1-\cos\tau_f)} + \frac{\Delta x_{2f}^2}{5\tau_f - 3\sin\tau_f} + \frac{i^2}{\tau_f + |\sin\tau_f|} \tag{A-127}$$

Rendezvous

$$\begin{aligned}\frac{J}{n_o^3 r_o^2} = {} & \frac{\frac{\Delta x_{1f}^2}{8}(5\tau_f + 3\sin\tau_f) - 4\,\Delta x_{1f}\,\Delta x_{3f}\sin\frac{\tau_f}{2} + \tau_f\,\Delta x_{3f}^2}{\tau_f(5\tau_f + 3\sin\tau_f) - 16(1-\cos\tau_f)} \\ & + \frac{\frac{1}{8}(5\tau_f - 3\sin\tau_f)\left\{2\,\Delta x_{2f}\cos\frac{\tau_f}{2} - 2\,\Delta x_{3f}\sin\frac{\tau_f}{2} - \Delta x_{4f} + \frac{3}{4}\tau_f\,\Delta x_{1f} - 4x_{30}\sin\frac{\tau_f}{2}\right\}^2}{\tau_f(5\tau_f - 3\sin\tau_f)\left(\frac{3}{16}\tau_f^2 + 1\right) - 2\left(3\tau_f\cos\frac{\tau_f}{2} - 8\sin\frac{\tau_f}{2}\right)^2} \\ & + \frac{\Delta x_{2f}\left(3\tau_f\cos\frac{\tau_f}{2} - 8\sin\frac{\tau_f}{2}\right)\left\{2\Delta x_{2f}\cos\frac{\tau_f}{2} - 2\Delta x_{3f}\sin\frac{\tau_f}{2} - \Delta x_{4f} + \frac{3}{4}\tau_f\,\Delta x_{1f} - 4x_{30}\sin\frac{\tau_f}{2}\right\}}{\tau_f(5\tau_f - 3\sin\tau_f)\left(\frac{3}{16}\tau_f^2 + 1\right) - 2\left(3\tau_f\cos\frac{\tau_f}{2} - 8\sin\frac{\tau_f}{2}\right)^2} \\ & + \frac{\tau_f\,\Delta x_{2f}^2\left(\frac{3}{16}\tau_f^2 + 1\right)}{\tau_f(5\tau_f - 3\sin\tau_f)\left(\frac{3}{16}\tau_f^2 + 1\right) - 2\left(3\tau_f\cos\frac{\tau_f}{2} - 8\sin\frac{\tau_f}{2}\right)^2} \\ & + i^2\left[\frac{\tau_f - \sin\tau_f\cos(2\Omega_f + \tau_f)}{(\tau_f^2 - \sin^2\tau_f)}\right]\end{aligned} \tag{A-128}$$

NOTE: (1) The second term of this equation is incorrect in Ref. 1.

10. The optimal values for changes in the state variables x_4 and Ω are predicted by the variational analysis in the case of orbit transfer where the values x_4 and Ω are left open at the final time.

$$\begin{aligned}\Delta x_4^* &= \frac{3}{4}\tau_f \Delta x_{1f} - 2\Delta x_{3f}\sin\frac{\tau_f}{2} - 4x_{30}\sin\frac{\tau_f}{2} \\ &+ \frac{4\Delta x_{2f}}{5\tau_f - 3\sin\tau_f}\left\{\frac{1}{2}\cos\frac{\tau_f}{2}(5\tau_f - 3\sin\tau_f) + 3\tau_f\cos\frac{\tau_f}{2} - 8\sin\frac{\tau_f}{2}\right\}\end{aligned} \tag{A-129}$$

$$\Omega_f^* = n\pi - \frac{\tau_f}{2} \tag{A-130}$$

11. Payoff Equations with an Intermediate Reference Orbit

Transfer

$$\begin{aligned}\frac{J}{n_I^3 r_I^2} &= \frac{\frac{\Delta x_{1f}^2}{8}(5\tau_f + 3\sin\tau_f) - 4\Delta x_{1f}\,\Delta x_{3f}\sin\frac{\tau_f}{2} + \tau_f\Delta x_{3f}^2}{\tau_f(5\tau_f + 3\sin\tau_f) - 16(1-\cos\tau_f)} + \frac{\Delta x_{2f}^2}{5\tau_f - 3\sin\tau_f} \\ &+ \frac{i^2}{\tau_f + |\sin\tau_f|}\end{aligned} \tag{A-131}$$

Rendezvous

$$\begin{aligned}\frac{J}{n_I^3 r_I^2} &= \frac{\frac{\Delta x_{1f}^2}{8}(5\tau_f + 3\sin\tau_f) - 4\Delta x_{1f}\,\Delta x_{3f}\sin\frac{\tau_f}{2} + \tau_f\Delta x_{3f}^2}{\tau_f(5\tau_f + 3\sin\tau_f) - 16(1-\cos\tau_f)} \\ &+ \frac{\frac{1}{8}(5\tau_f - 3\sin\tau_f)\left\{2\Delta x_{2f}\cos\frac{\tau_f}{2} - 2\Delta x_{3f}\sin\frac{\tau_f}{2} - \Delta x_{4f} + \frac{3}{4}\tau_f(x_{10} + x_{1f} - 2) - 4x_{30}\sin\frac{\tau_f}{2}\right\}^2}{\tau_f(5\tau_f - 3\sin\tau_f)(\frac{3}{16}\tau_f^2 + 1) - 2(3\tau_f\cos\frac{\tau_f}{2} - 8\sin\frac{\tau_f}{2})^2} \\ &+ \frac{\Delta x_{2f}(3\tau_f\cos\frac{\tau_f}{2} - 8\sin\frac{\tau_f}{2})\left\{2\Delta x_{2f}\cos\frac{\tau_f}{2} - 2\Delta x_{3f}\sin\frac{\tau_f}{2} - \Delta x_{4f} + \frac{3}{4}\tau_f(x_{10} + x_{1f} - 2) - 4x_{30}\sin\frac{\tau_f}{2}\right\}}{\tau_f(5\tau_f - 3\sin\tau_f)(\frac{3}{16}\tau_f^2 + 1) - 2(3\tau_f\cos\frac{\tau_f}{2} - 8\sin\frac{\tau_f}{2})^2} \\ &+ \frac{\tau_f\Delta x_{2f}^2(\frac{3}{16}\tau_f^2 + 1)}{\tau_f(5\tau_f - 3\sin\tau_f)(\frac{3}{16}\tau_f^2 + 1) - 2(3\tau_f\cos\frac{\tau_f}{2} - 8\sin\frac{\tau_f}{2})^2} \\ &+ i^2\left[\frac{\tau_f - \sin\tau_f\cos(2\Omega_f + \tau_f)}{\tau_f^2 - \sin^2\tau_f}\right]\end{aligned} \tag{A-132}$$

12. Optimal Transfer Coordinates

$$\Delta x_4{}^{*} = \frac{3}{4}\tau_f(x_{10} + x_{1f} - 2) - 2\Delta x_{3f}\sin\frac{\tau_f}{2} - 4x_{30}\sin\frac{\tau_f}{2} + \frac{4\,\Delta x_{2f}}{5\tau_f - 3\sin\tau_f}\left\{\frac{1}{2}\cos\frac{\tau_f}{2}(5\tau_f - 3\sin\tau_f) + 3\tau_f\cos\frac{\tau_f}{2} - 8\sin\frac{\tau_f}{2}\right\} \tag{A-133}$$

$$\Omega_f{}^{*} = n\pi - \frac{\tau_f}{2} \tag{A-134}$$

APPENDIX III

SYNTHESIS OF THE OPTIMAL CONTROLS

A. Rotating Coordinates

1. Control Equations

$$A_y = \frac{\partial A_y}{\partial y} y + \frac{\partial A_y}{\partial u} u + \frac{\partial A_y}{\partial v'} v + \frac{\partial A_y}{\partial x} x \tag{A-135}$$

$$A_x = \frac{\partial A_x}{\partial y} y + \frac{\partial A_x}{\partial u} u + \frac{\partial A_x}{\partial v} v + \frac{\partial A_x}{\partial x} x \tag{A-136}$$

$$A_z = \frac{\partial A_z}{\partial z} z + \frac{\partial A_z}{\partial w} w \tag{A-137}$$

2. Guidance Coefficients - Transfer

$$\frac{\partial A_y}{\partial y} = \frac{12\tau'}{\Phi} (1 - \cos\tau')(29 - 27\cos\tau') \tag{A-138}$$

$$\frac{\partial A_y}{\partial u} = \frac{24}{\Phi} (1 - \cos\tau')(11\sin\tau' - 3\tau'\cos\tau' - 8\tau') \tag{A-139}$$

$$\frac{\partial A_y}{\partial v} = \frac{12}{\Phi} (5\tau'^2 + 3\tau'\sin\tau'\cos\tau' - 8\sin^2\tau') \tag{A-140}$$

$$\frac{\partial A_x}{\partial y} = \frac{12}{\Phi} \left[70\tau'\sin\tau' - 55\tau'^2 + 18\tau'\sin\tau'\cos\tau' + 3(1 - \cos\tau')(5 - 27\cos\tau') \right] \tag{A-141}$$

$$\frac{\partial A_x}{\partial u} = \frac{6}{\Phi} \left[65\tau'^2 - 80\tau'\sin\tau' - 24\tau'\sin\tau'\cos\tau' - (1 - \cos\tau')(25 - 103\cos\tau') \right] \tag{A-142}$$

$$\frac{\partial A_x}{\partial v} = -\frac{24}{\Phi} (8\tau' - 11\sin\tau' + 3\tau'\cos\tau')(1 - \cos\tau')^{(1)} \tag{A-143}$$

(1) NOTE : $\frac{\partial A_x}{\partial v} = \frac{\partial A_y}{\partial u}$

$$\frac{\partial A_z}{\partial z} = \frac{-2\sin^2\tau'}{\tau'^2 - \sin^2\tau'} \tag{A-144}$$

$$\frac{\partial A_z}{\partial w} = \frac{-(2\tau' - \sin 2\tau')}{\tau'^2 - \sin^2\tau'} \tag{A-145}$$

where

$$\Phi = 480\tau' - 75\tau'^3 - 240\tau'\cos\tau'(1+\cos\tau') - 144\sin\tau'(1-\cos\tau') - 213\tau'\sin^2\tau' \tag{A-146}$$

3. Rendezvous

Due to the length and complexity of the synthesized, in-plane, control equations for rendezvous, the guidance coefficients are not written explicitly here. Instead the basic equations are tabulated, and the coefficients calculated from these equations are plotted in Figs. 20 through 22.

$$\frac{\partial A_x}{\partial x_i} = 3\frac{\partial C_4}{\partial x_i} - 3\frac{\partial C_0}{\partial x_i}\tau' - 4\frac{\partial C_1}{\partial x_i}\cos\tau' - 4\frac{\partial C_2}{\partial x_i}\sin\tau' \tag{A-147}$$

$$\frac{\partial A_y}{\partial x_i} = 2\left(\frac{\partial C_0}{\partial x_i} - \frac{\partial C_1}{\partial x_i}\sin\tau' + \frac{\partial C_2}{\partial x_i}\cos\tau'\right) \tag{A-148}$$

$$A_z = \frac{\partial A_z}{\partial z} z + \frac{\partial A_z}{\partial w} w \tag{A-149}$$

$$C_0 = \frac{\begin{vmatrix} x & \phi_{11} & \phi_{12} & \phi_{14} \\ y & \phi_{21} & \phi_{22} & \phi_{24} \\ u & \phi_{31} & \phi_{32} & \phi_{34} \\ v & \phi_{41} & \phi_{42} & \phi_{44} \end{vmatrix}}{D} \tag{A-150}$$

$$C_1 = \frac{\begin{vmatrix} \phi_{10} & x & \phi_{12} & \phi_{14} \\ \phi_{20} & y & \phi_{22} & \phi_{24} \\ \phi_{30} & u & \phi_{32} & \phi_{34} \\ \phi_{40} & v & \phi_{42} & \phi_{44} \end{vmatrix}}{D} \tag{A-151}$$

$$C_2 = \frac{\begin{vmatrix} \phi_{10} & \phi_{11} & x & \phi_{14} \\ \phi_{20} & \phi_{21} & y & \phi_{24} \\ \phi_{30} & \phi_{31} & u & \phi_{34} \\ \phi_{40} & \phi_{41} & v & \phi_{44} \end{vmatrix}}{D} \qquad (A-152)$$

$$C_4 = \frac{\begin{vmatrix} \phi_{10} & \phi_{11} & \phi_{12} & x \\ \phi_{20} & \phi_{21} & \phi_{22} & y \\ \phi_{30} & \phi_{31} & \phi_{32} & u \\ \phi_{40} & \phi_{41} & \phi_{42} & v \end{vmatrix}}{D} \qquad (A-153)$$

where

$$D = \begin{vmatrix} \phi_{10} & \phi_{11} & \phi_{12} & \phi_{14} \\ \phi_{20} & \phi_{21} & \phi_{22} & \phi_{24} \\ \phi_{30} & \phi_{31} & \phi_{32} & \phi_{34} \\ \phi_{40} & \phi_{41} & \phi_{42} & \phi_{44} \end{vmatrix} \qquad (A-154)$$

and

$$\phi_{10} = \frac{3}{4}\tau'^3 - 8\tau' + 8\sin\tau' \qquad \phi_{30} = 8(1-\cos\tau') - \frac{9}{4}\tau'^2$$

$$\phi_{11} = 8(1-\cos\tau') - 5\tau'\sin\tau' \qquad \phi_{31} = 5\tau'\cos\tau' - 3\sin\tau'$$

$$\phi_{12} = 5\tau'\cos\tau' - 11\sin\tau' + 6\tau' \qquad \phi_{32} = 5\tau'\sin\tau' - 6(1-\cos\tau')$$

$$\phi_{14} = 6(1-\cos\tau') - \frac{9}{4}\tau'^2 \qquad \phi_{34} = \frac{9}{2}\tau' - 6\sin\tau'$$

$$\phi_{20} = 4(1-\cos\tau') - \frac{3}{2}\tau'^2 \qquad \phi_{40} = 3\tau' - 4\sin\tau' \qquad (A-155)$$

$$\phi_{21} = \frac{5}{2}\left[\tau'\cos\tau' - \sin\tau'\right] \qquad \phi_{41} = \frac{5}{2}\tau'\sin\tau'$$

$$\phi_{22} = \frac{5}{2}\tau'\sin\tau' - 4(1-\cos\tau') \qquad \phi_{42} = \frac{3}{2}\sin\tau' - \frac{5}{2}\tau'\cos\tau'$$

$$\phi_{24} = 3(\tau' - \sin\tau') \qquad \phi_{44} = -3(1-\cos\tau')$$

$$\frac{\partial A_z}{\partial z} = \frac{-2\sin^2\tau'}{\tau'^2 - \sin^2\tau'} \tag{A-156}$$

$$\frac{\partial A_z}{\partial w} = \frac{-(2\tau' - \sin 2\tau')}{\tau'^2 - \sin^2\tau'} \tag{A-157}$$

B. Lagrange Variables

1. Control Equations

$$A_R = \frac{\partial A_R}{\partial \Delta x_1}\Delta x_1 + \frac{\partial A_R}{\partial \Delta x_2}\Delta x_2 + \frac{\partial A_R}{\partial \Delta x_3}\Delta x_3 + \frac{\partial A_R}{\partial \Delta x_4}\Delta x_4 + \frac{\partial A_R}{\partial x_{30}} x_{30} \tag{A-158}$$

$$A_S = \frac{\partial A_S}{\partial \Delta x_1}\Delta x_1 + \frac{\partial A_S}{\partial \Delta x_2}\Delta x_2 + \frac{\partial A_S}{\partial \Delta x_3}\Delta x_3 + \frac{\partial A_S}{\partial \Delta x_4}\Delta x_4 + \frac{\partial A_S}{\partial x_{30}} x_{30} \tag{A-159}$$

$$A_W = \frac{\partial A_W}{\partial \Delta x_5}\Delta x_5 + \frac{\partial A_W}{\partial \Delta x_6}\Delta x_6 \tag{A-160}$$

2. Guidance Coefficients - Transfer

$$\frac{\partial A_R}{\partial \Delta x_1} = \frac{-4\sin\tau' \sin\frac{\tau'}{2}}{\tau'(5\tau' + 3\sin\tau') - 16(1 - \cos\tau')} \tag{A-161}$$

$$\frac{\partial A_R}{\partial \Delta x_2} = \frac{2\cos\tau'}{5\tau' - 3\sin\tau'} \tag{A-162}$$

$$\frac{\partial A_R}{\partial \Delta x_3} = \frac{2\tau'\sin\tau'}{\tau'(5\tau' + 3\sin\tau') - 16(1 - \cos\tau')} \tag{A-163}$$

$$\frac{\partial A_S}{\partial \Delta x_1} = \frac{8\cos\tau'\sin\frac{\tau'}{2} - \frac{1}{2}(5\tau' + 3\sin\tau')}{\tau'(5\tau' + 3\sin\tau') - 16(1 - \cos\tau')} \tag{A-164}$$

$$\frac{\partial A_S}{\partial \Delta x_2} = \frac{4\sin\tau'}{5\tau' - 3\sin\tau'} \tag{A-165}$$

$$\frac{\partial A_W}{\partial \Delta x_5} = \frac{\cos\tau' \sin^2\tau'}{\tau'^2 - \sin^2\tau'} \tag{A-166}$$

$$\frac{\partial A_W}{\partial \Delta x_6} = \frac{-\frac{1}{2}\cos\tau'(2\tau' - \sin 2\tau')}{\tau'^2 - \sin^2\tau'} \tag{A-167}$$

$$\frac{\partial A_S}{\partial \Delta x_3} = \frac{4(2\sin\frac{\tau'}{2} - \tau'\cos\tau')}{\tau'(5\tau' + 3\sin\tau') - 16(1 - \cos\tau')} \tag{A-168}$$

3. Guidance Coefficients - Rendezvous

$$\frac{\partial A_R}{\partial \Delta x_1} = \frac{4\sin\tau'\sin\frac{\tau'}{2}}{Q} - \frac{\frac{3}{8}\tau\left[5\tau' - 3\sin\tau' + 2\cos\tau'(3\tau'\cos\frac{\tau'}{2} - 8\sin\frac{\tau'}{2})\right]}{B} \tag{A-169}$$

$$\frac{\partial A_R}{\partial \Delta x_2} = \frac{2\tau'\cos\tau'(\frac{3}{16}\tau'^2 + 1) + \cos\frac{\tau'}{2}(5\tau' - 3\sin\tau') + 2(3\tau'\cos\frac{\tau'}{2} - 8\sin\frac{\tau'}{2})(1 + \cos\tau'\cos\frac{\tau'}{2})}{B} \tag{A-170}$$

$$\frac{\partial A_R}{\partial \Delta x_3} = \frac{-2\tau'\sin\tau'}{Q} + \frac{\sin\frac{\tau'}{2}\left[5\tau' - 3\sin\tau' + 2\cos\tau'(3\tau'\cos\frac{\tau'}{2} - 8\sin\frac{\tau'}{2})\right]}{B} \tag{A-171}$$

$$\frac{\partial A_R}{\partial \Delta x_4} = -\frac{\frac{1}{2}\left[5\tau' - 3\sin\tau' + 2\cos\tau'(3\tau'\cos\frac{\tau'}{2} - 8\sin\frac{\tau'}{2})\right]}{B} \tag{A-172}$$

$$\frac{\partial A_R}{\partial x_{30}} = \frac{2\sin\frac{\tau'}{2}\left[5\tau' - 3\sin\tau' + 2\cos\tau'(3\tau'\cos\frac{\tau'}{2} - 8\sin\frac{\tau'}{2})\right]}{B} \tag{A-173}$$

$$\frac{\partial A_S}{\partial \Delta x_1} = \frac{\frac{1}{2}(5\tau' + 3\sin\tau' - 16\sin\frac{\tau'}{2}\cos\tau')}{Q} - \frac{\frac{3}{16}\tau'\left[3\tau'(5\tau' - 3\sin\tau') + 8\sin\tau'(3\tau'\cos\frac{\tau'}{2} - 8\sin\frac{\tau'}{2})\right]}{B} \tag{A-174}$$

$$\frac{\partial A_S}{\partial \Delta x_2} = \frac{4\tau' \sin\tau'\left(\frac{3}{16}\tau'^2 + 1\right) + \frac{3}{2}\tau' \cos\frac{\tau'}{2}\left(5\tau' - 3\sin\tau'\right)}{B} + \frac{\left(3\tau' \cos\frac{\tau'}{2} - 8\sin\frac{\tau'}{2}\right)\left(3\tau' + 4\sin\tau' \cos\frac{\tau'}{2}\right)}{B} \tag{A-175}$$

$$\frac{\partial A_S}{\partial \Delta x_3} = \frac{4\left(\tau' \cos\tau' - 2\sin\frac{\tau'}{2}\right)}{Q} + \frac{\frac{1}{2}\sin\frac{\tau'}{2}\left[3\tau'\left(5\tau' - 3\sin\tau'\right) + 8\sin\tau'\left(3\tau' \cos\frac{\tau'}{2} - 8\sin\frac{\tau'}{2}\right)\right]}{B} \tag{A-176}$$

$$\frac{\partial A_S}{\partial \Delta x_4} = -\frac{\frac{1}{4}\left[3\tau'\left(5\tau' - 3\sin\tau'\right) + 8\sin\tau'\left(3\tau' \cos\frac{\tau'}{2} - 8\sin\frac{\tau'}{2}\right)\right]}{B} \tag{A-177}$$

$$\frac{\partial A_S}{\partial x_{30}} = \frac{\sin\frac{\tau'}{2}\left[3\tau'\left(5\tau' - 3\sin\tau'\right) + 8\sin\tau'\left(3\tau' \cos\frac{\tau'}{2} - 8\sin\frac{\tau'}{2}\right)\right]}{B} \tag{A-178}$$

$$\frac{\partial A_W}{\partial \Delta x_5} = \frac{2\sin\tau'\left(\tau' + \sin 2\tau'\right)}{\tau'^2 - \sin^2\tau'} \tag{A-179}$$

$$\frac{\partial A_W}{\partial \Delta x_6} = -\frac{2\left[\sin\tau' + \cos\tau'\left(\tau' - \sin 2\tau'\right)\right]}{\tau'^2 - \sin^2\tau'} \tag{A-180}$$

where

$$Q = 16\left(1 - \cos\tau'\right) - \tau'\left(5\tau' + 3\sin\tau'\right) \tag{A-181}$$

$$B = \tau'\left(5\tau' - 3\sin\tau'\right)\left(\frac{3}{16}\tau'^2 + 1\right) - 2\left(8\sin\frac{\tau'}{2} - 3\tau' \cos\frac{\tau'}{2}\right)^2 \tag{A-182}$$

RECTANGULAR COORDINATE SYSTEM

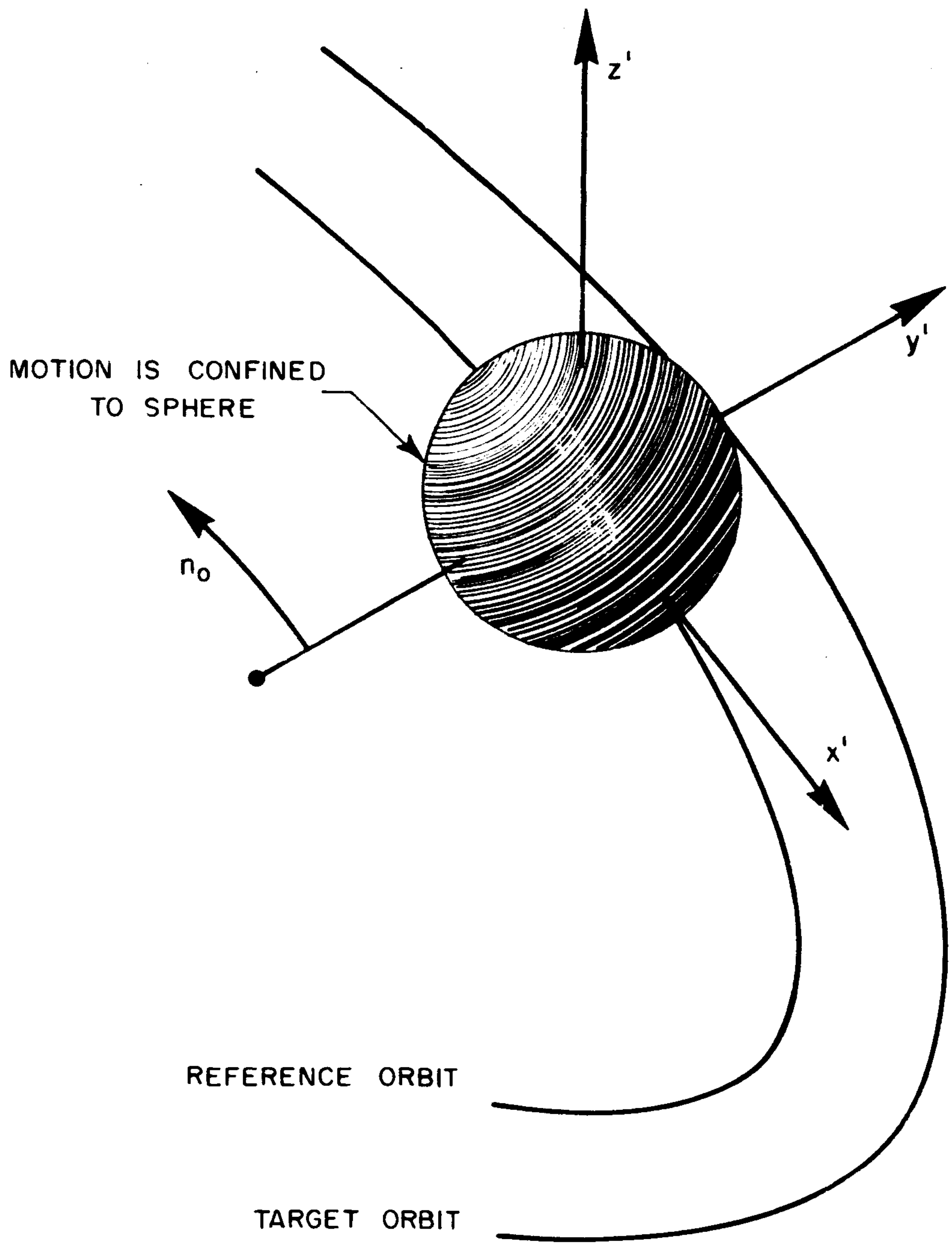

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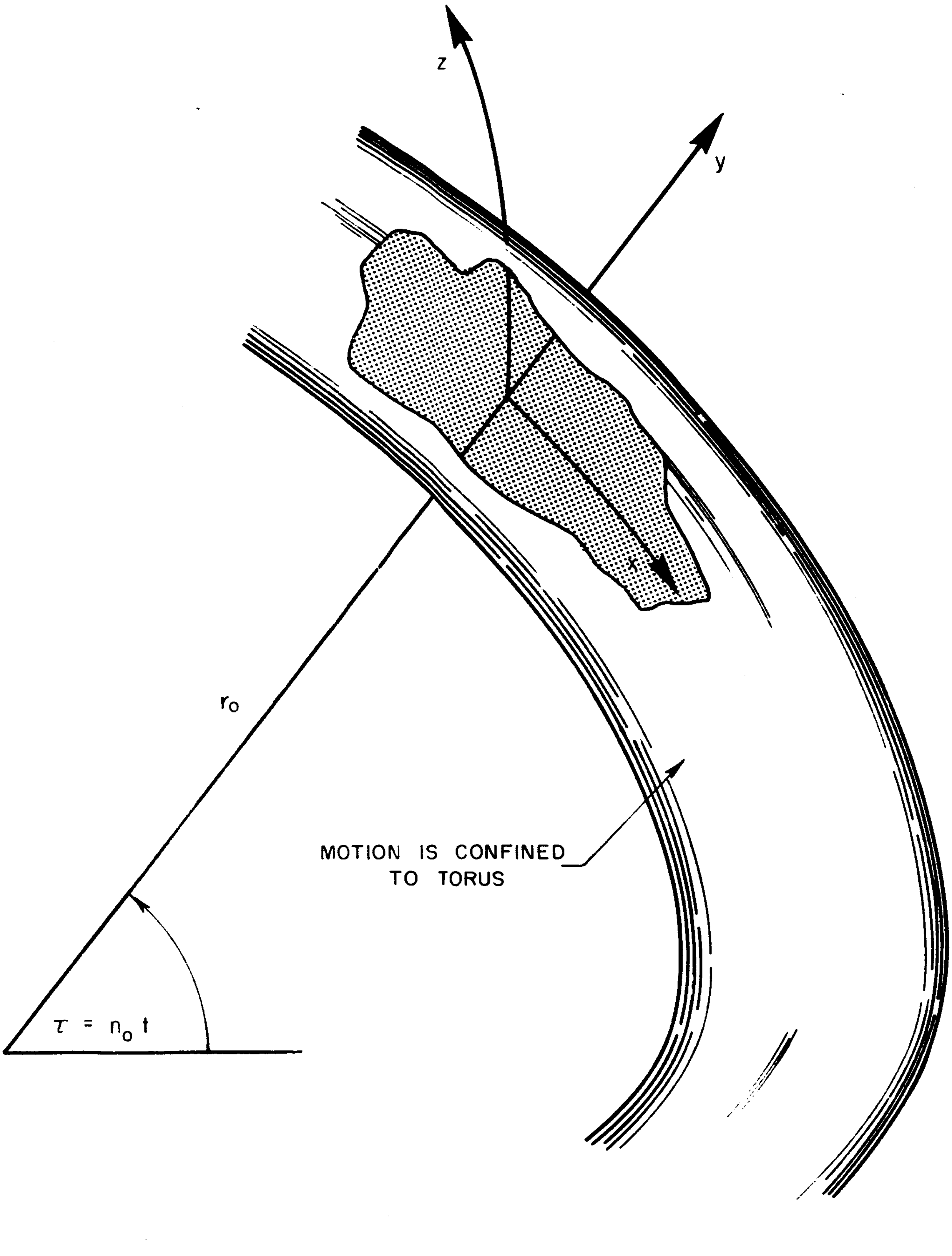
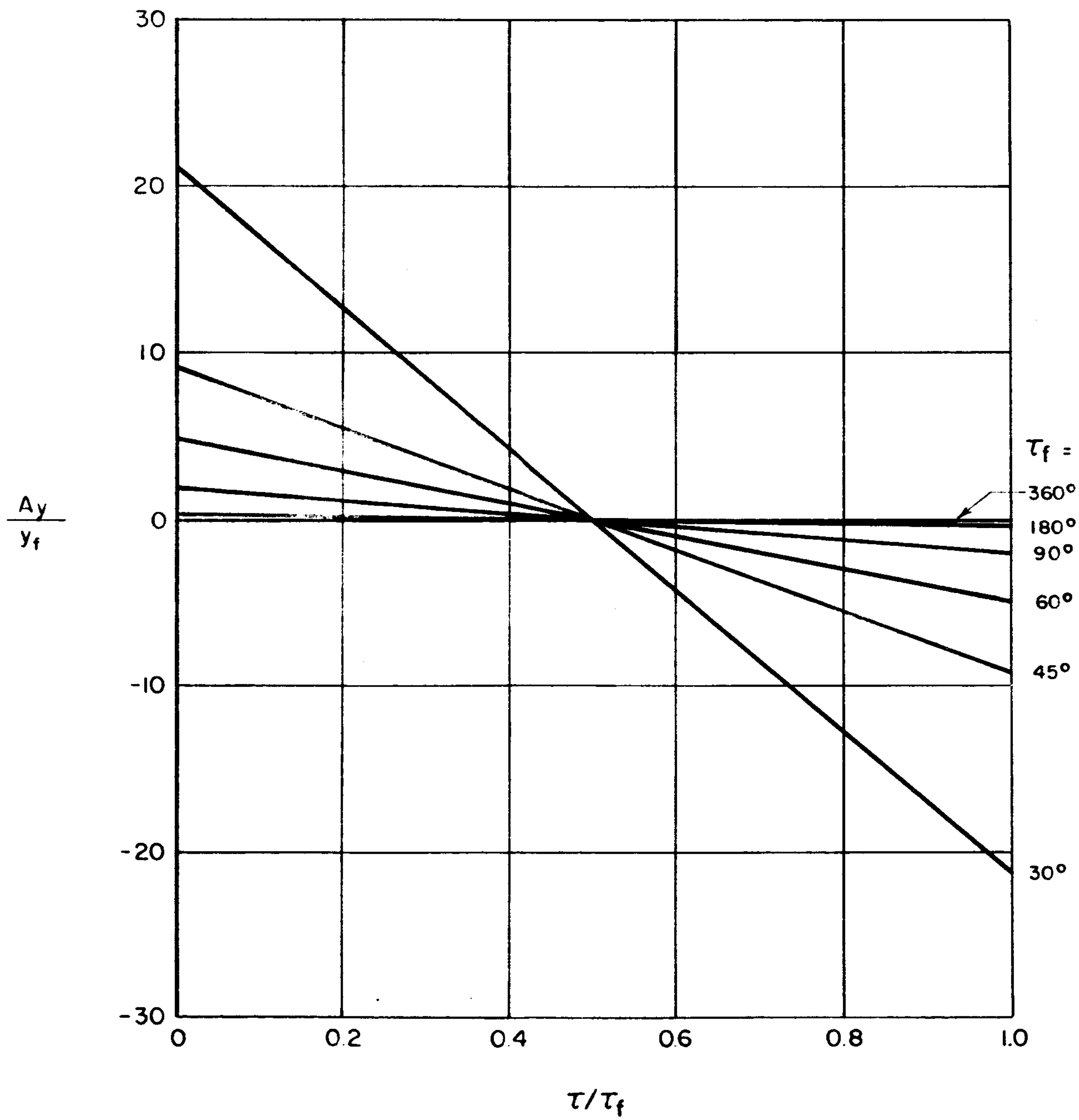
FIG. 2

SPHERICAL COORDINATE SYSTEM



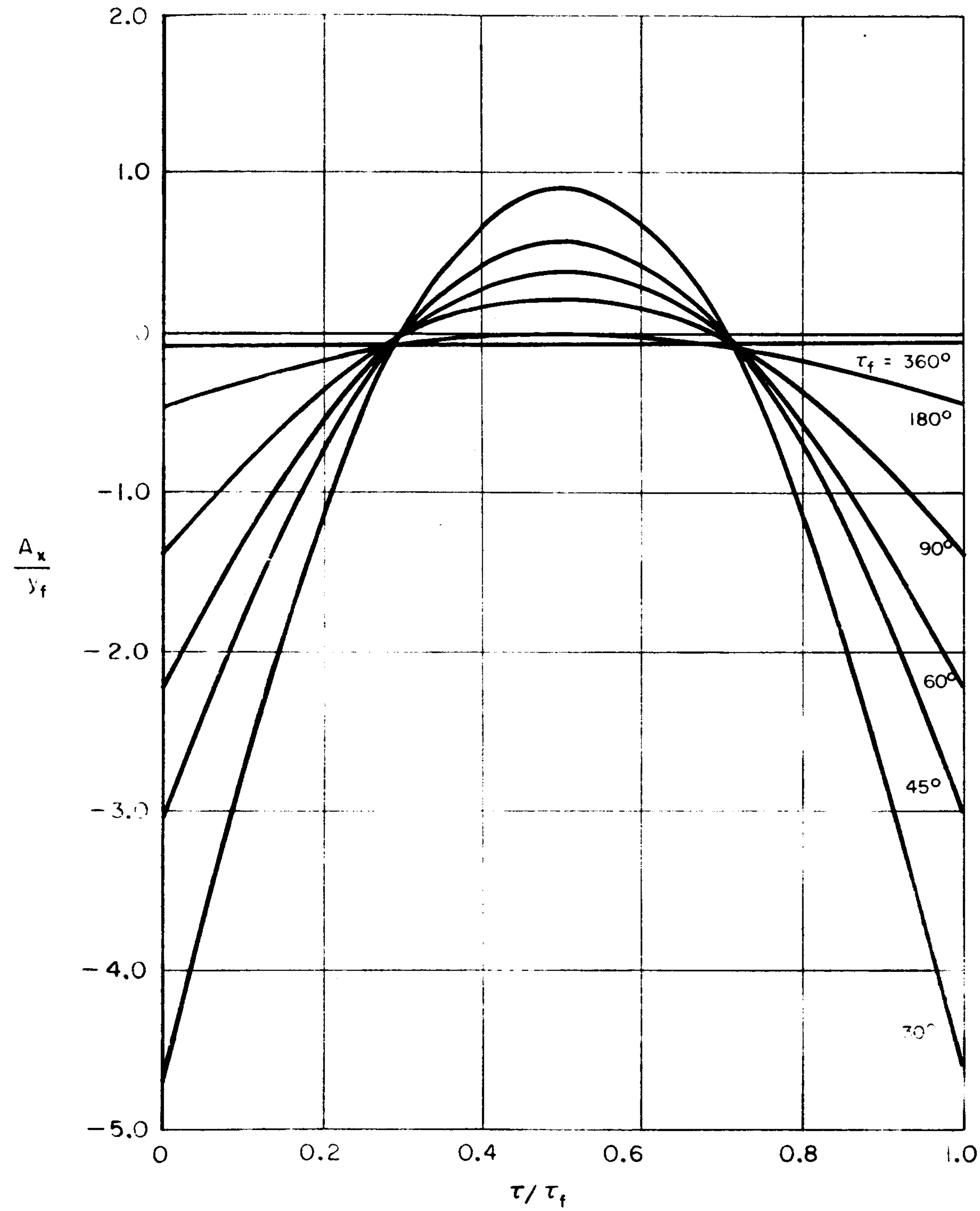
RADIAL ACCELERATION

CIRCLE-TO-CIRCLE TRANSFER



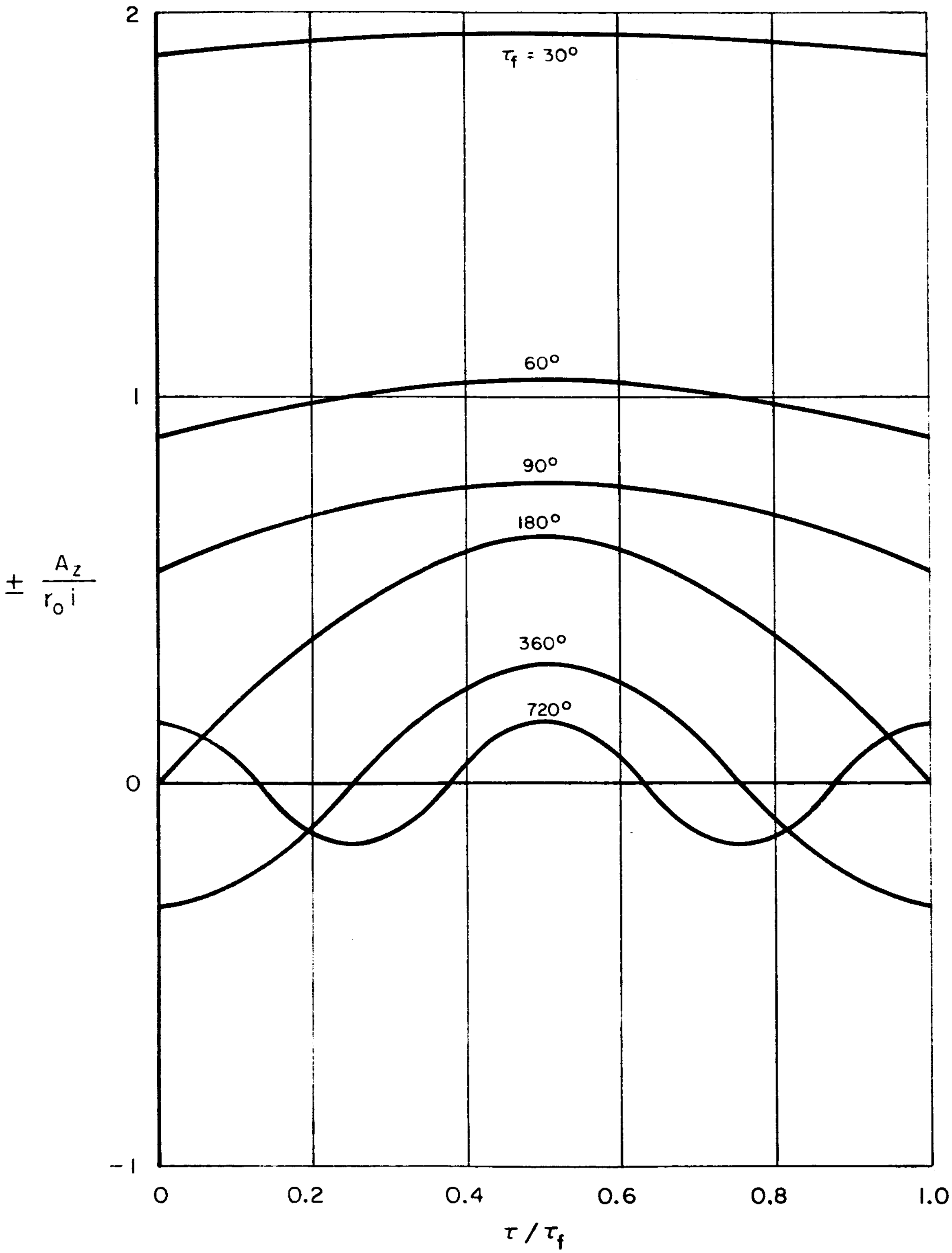
CIRCUMFERENTIAL ACCELERATION

CIRCLE - TO - CIRCLE TRANSFER



NORMAL ACCELERATION

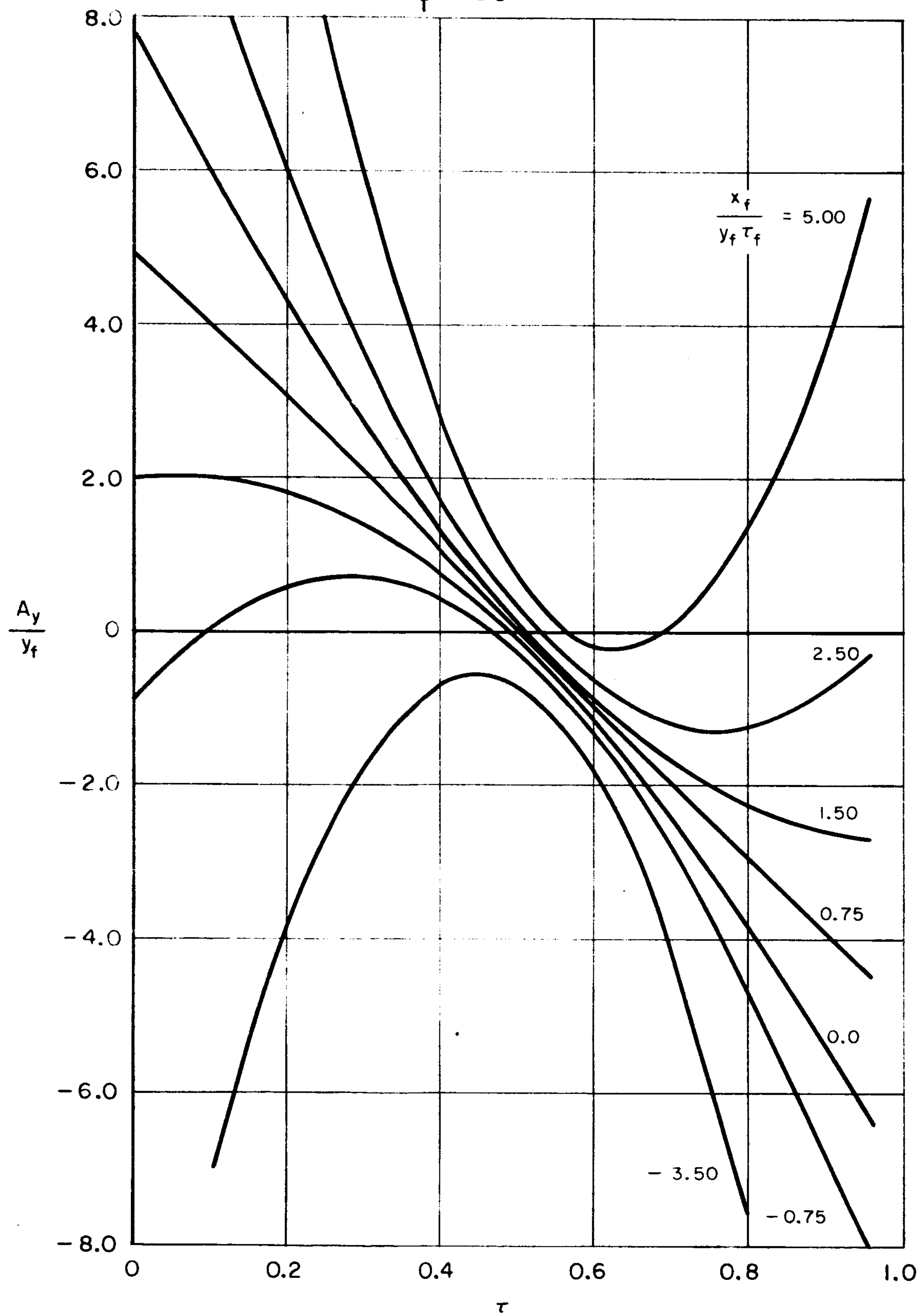
CIRCLE-TO-CIRCLE TRANSFER



RADIAL ACCELERATION

CIRCLE - TO - CIRCLE RENDEZVOUS

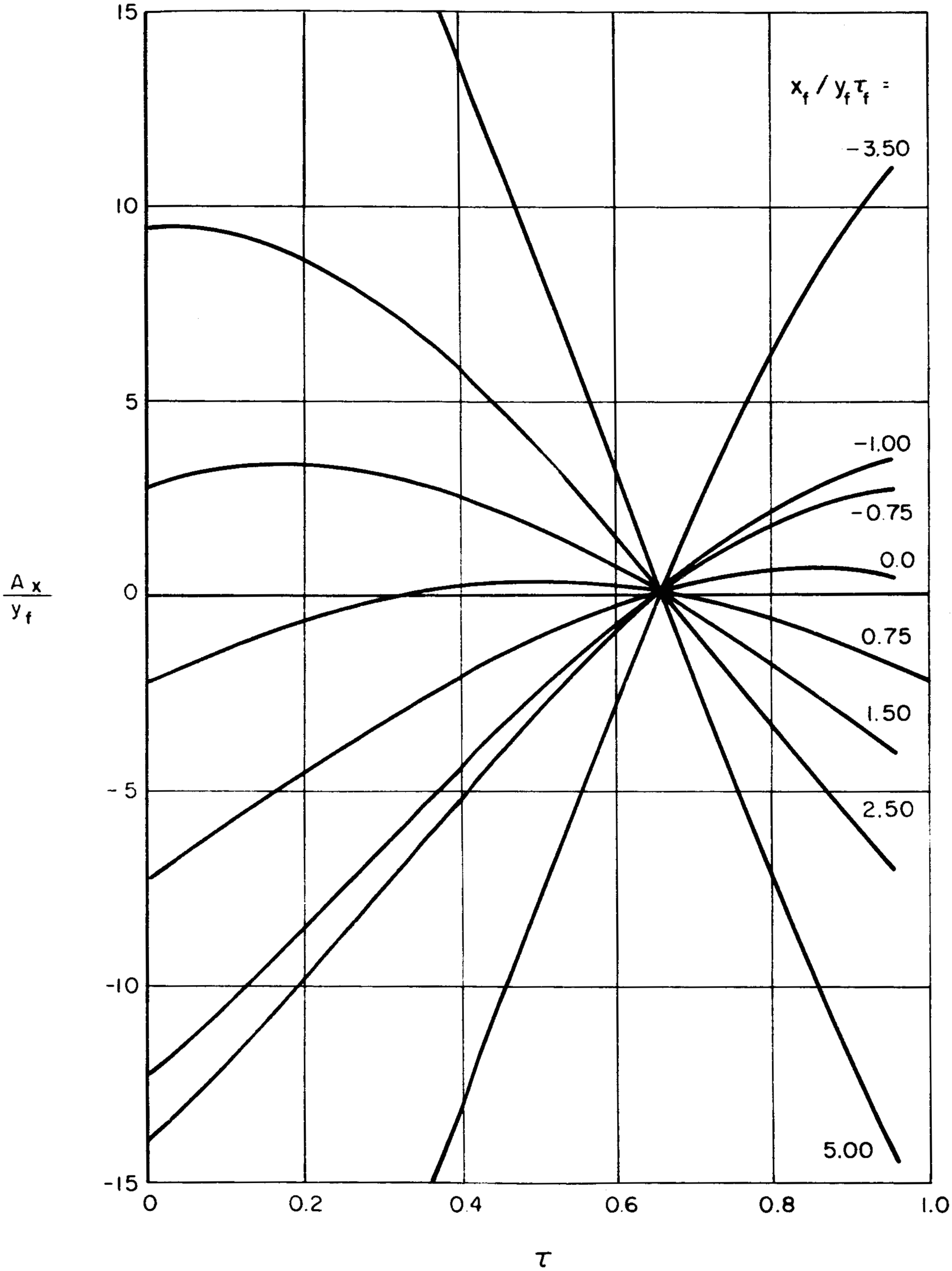
$\tau_f = 60°$



CIRCUMFERENTIAL ACCELERATION

CIRCLE-TO-CIRCLE RENDEZVOUS

$\tau_f = 60°$

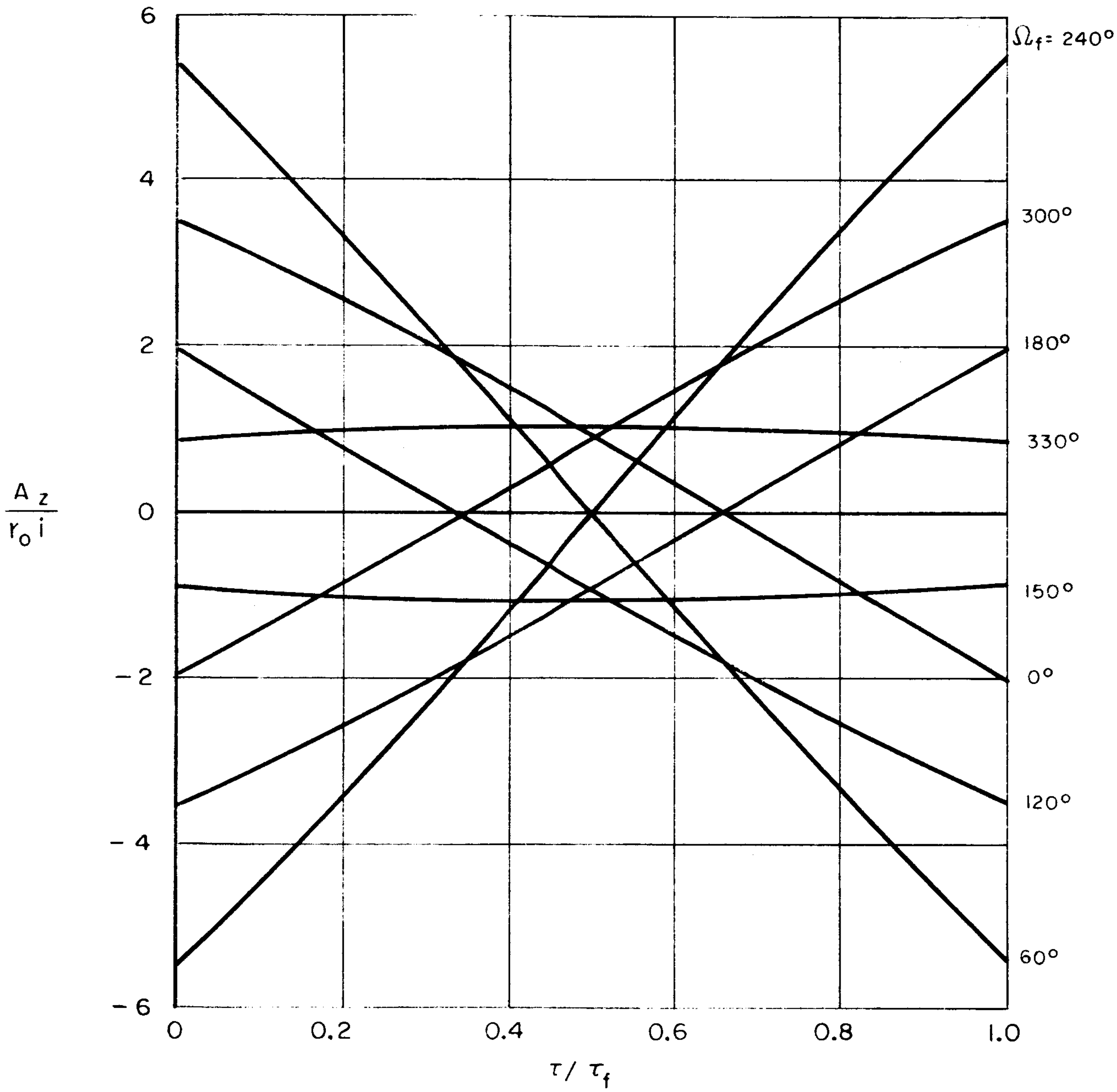


NORMAL ACCELERATION

CIRCLE - TO - CIRCLE RENDEZVOUS

$\tau_f = 60°$

OPTIMUM TRANSFERS : $\Omega_f = 150°, 330°$



IN-PLANE COMPONENT OF J
CIRCLE-TO-CIRCLE TRANSFER

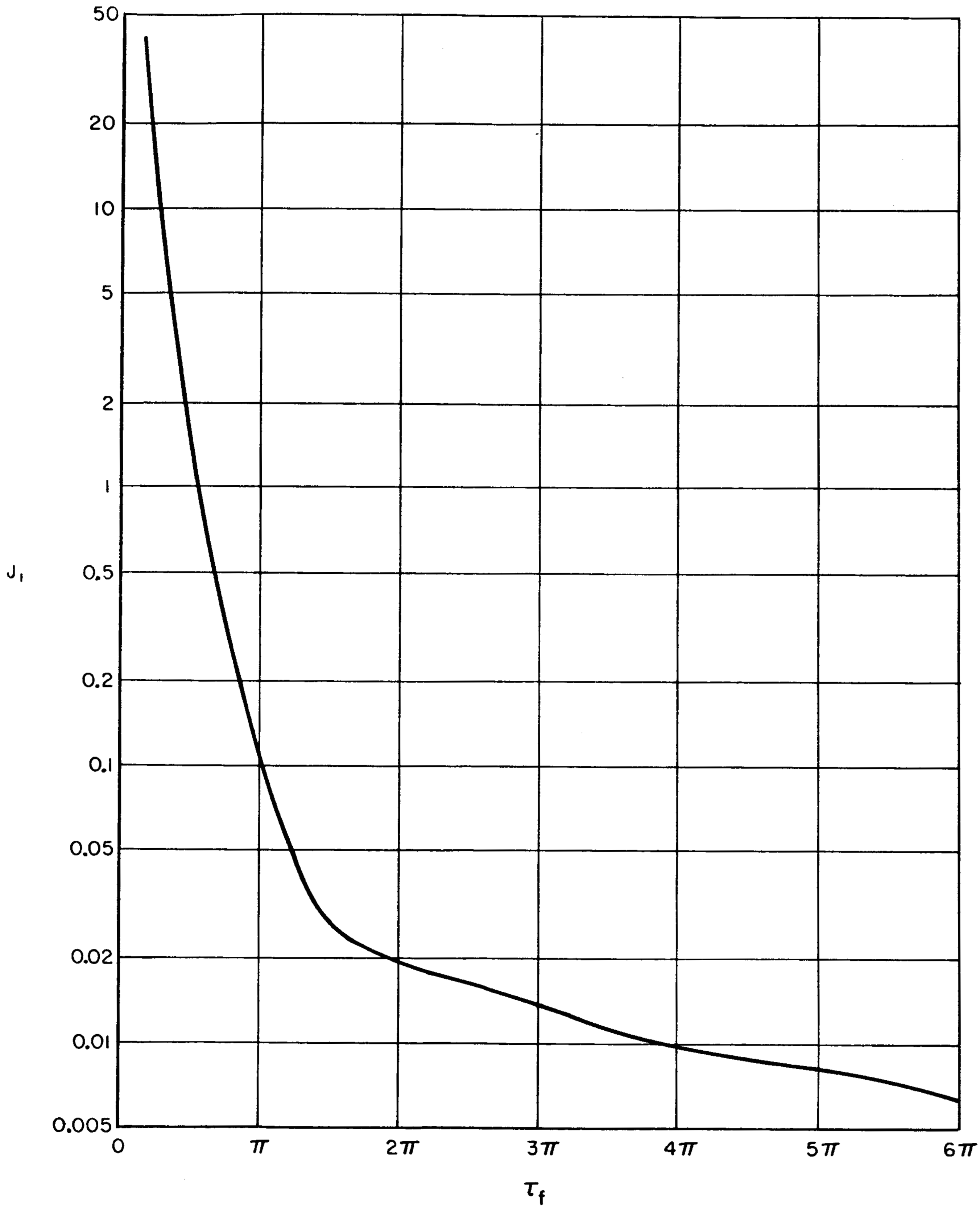
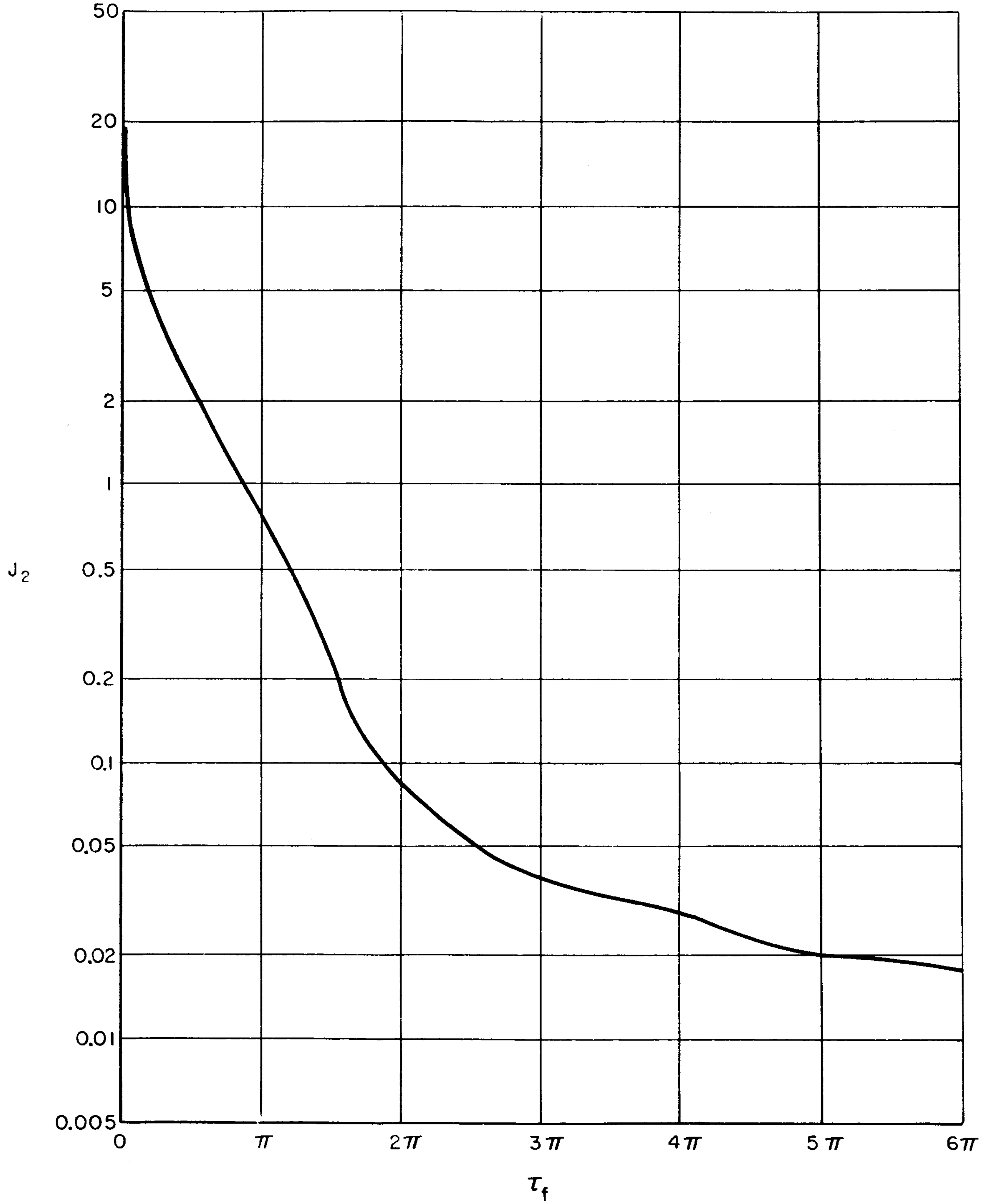


FIG. 10

IN-PLANE COMPONENT OF J

CIRCLE-TO-CIRCLE RENDEZVOUS



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OUT-OF-PLANE COMPONENT OF J

CIRCLE-TO-CIRCLE RENDEZVOUS

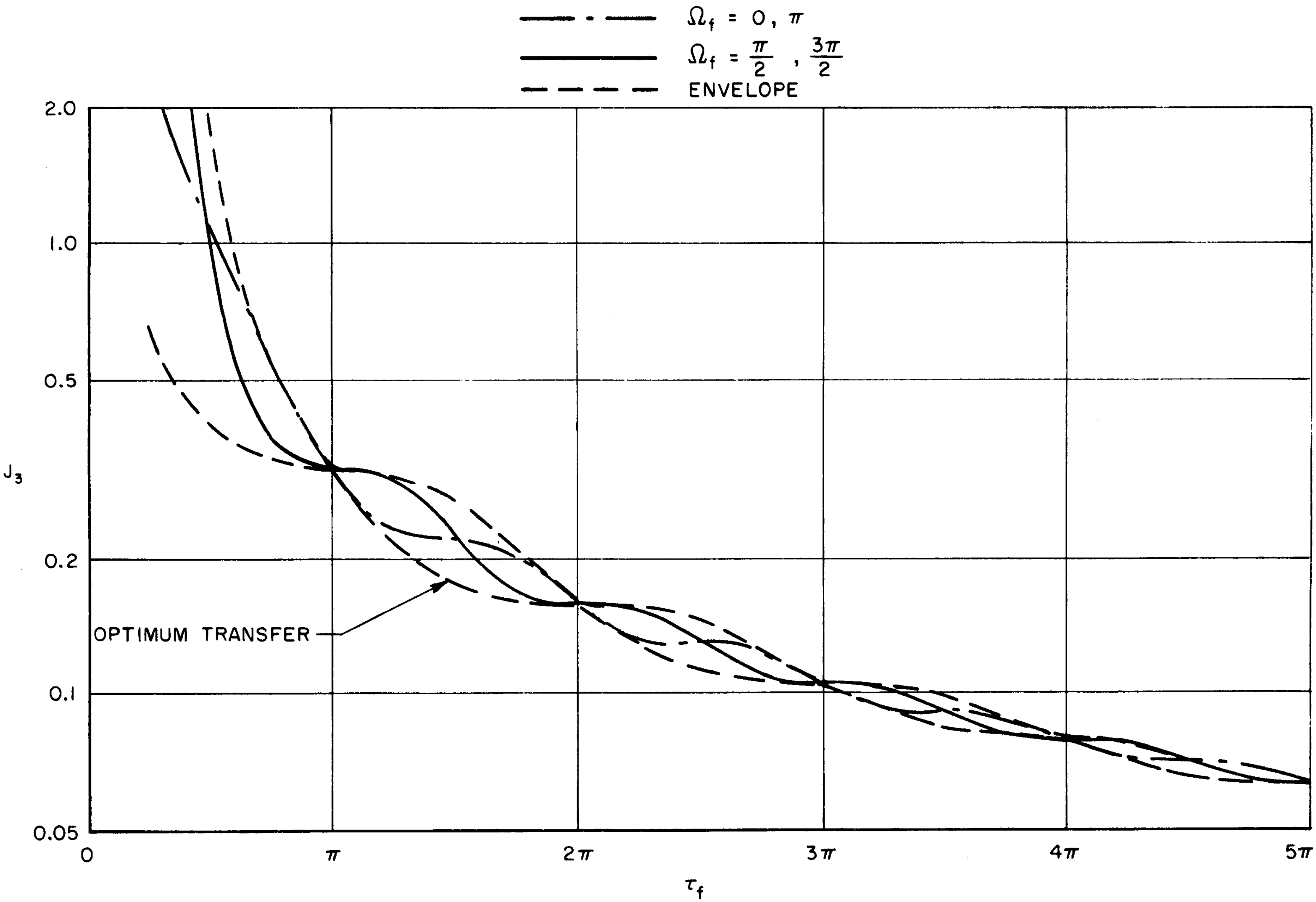
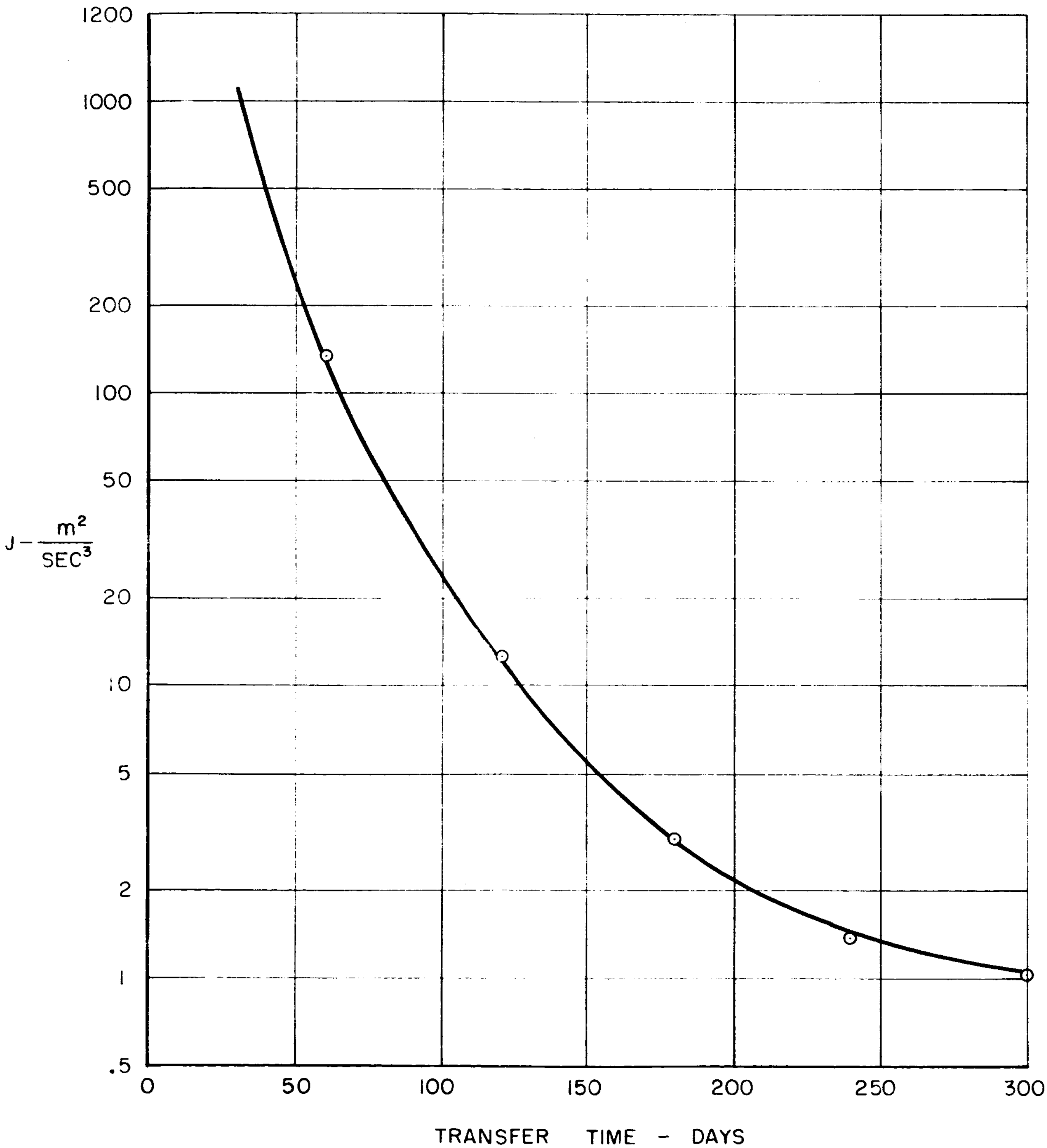


FIG. 11

OPTIMUM EARTH-VENUS TRANSFER

UNINCLINED CIRCULAR TERMINAL ORBITS

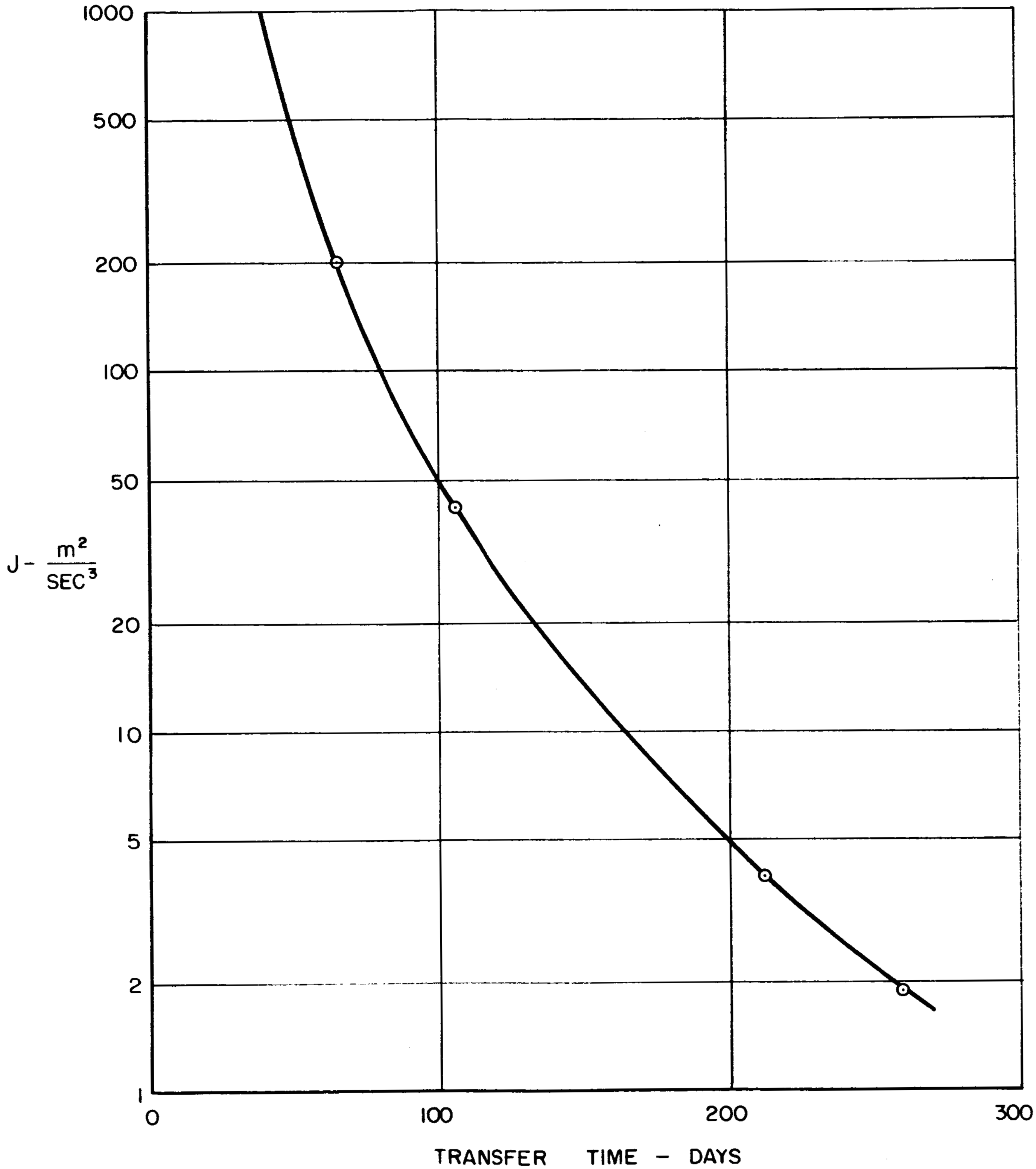
⊙ – LINEARIZED ANALYSIS



OPTIMUM EARTH - MARS TRANSFER

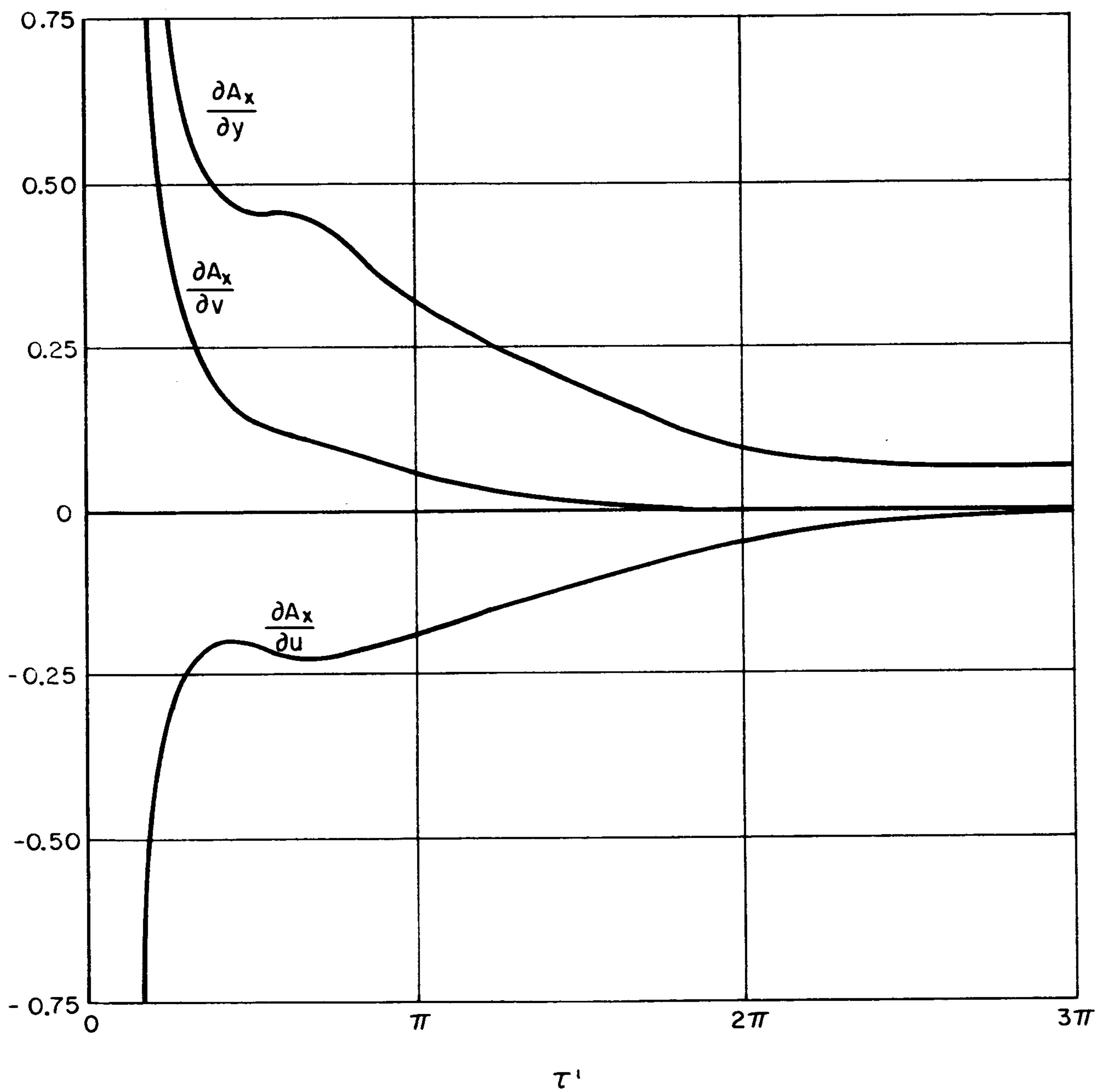
UNINCLINED CIRCULAR TERMINAL ORBITS

⊙ - LINEARIZED ANALYSIS



GUIDANCE COEFFICIENTS FOR OPTIMUM CONTROL ORBIT TRANSFER

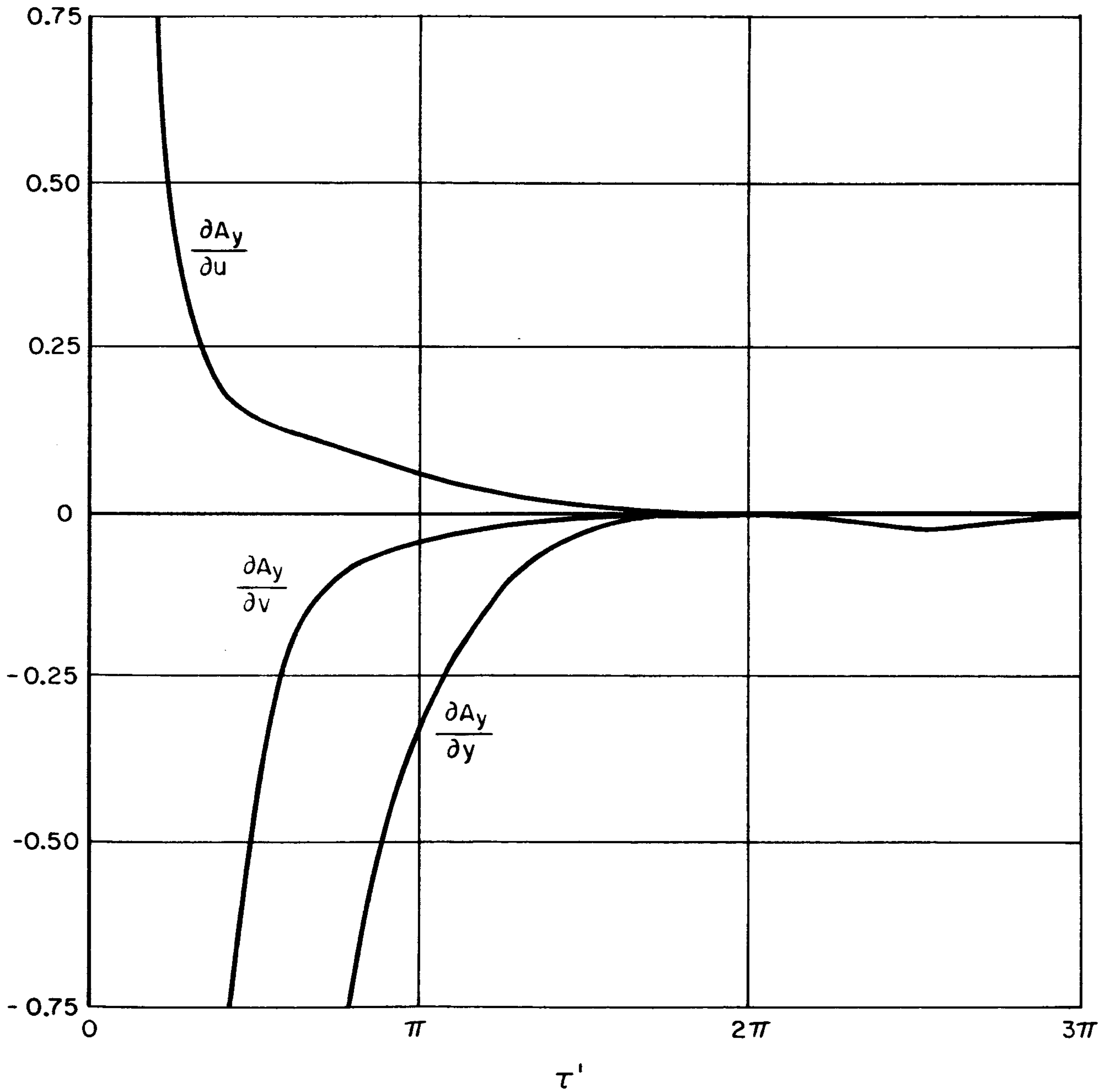
$$A_x = \frac{\partial A_x}{\partial y}\, y + \frac{\partial A_x}{\partial u}\, u + \frac{\partial A_x}{\partial v}\, v$$



GUIDANCE COEFFICIENTS FOR OPTIMUM CONTROL

ORBIT TRANSFER

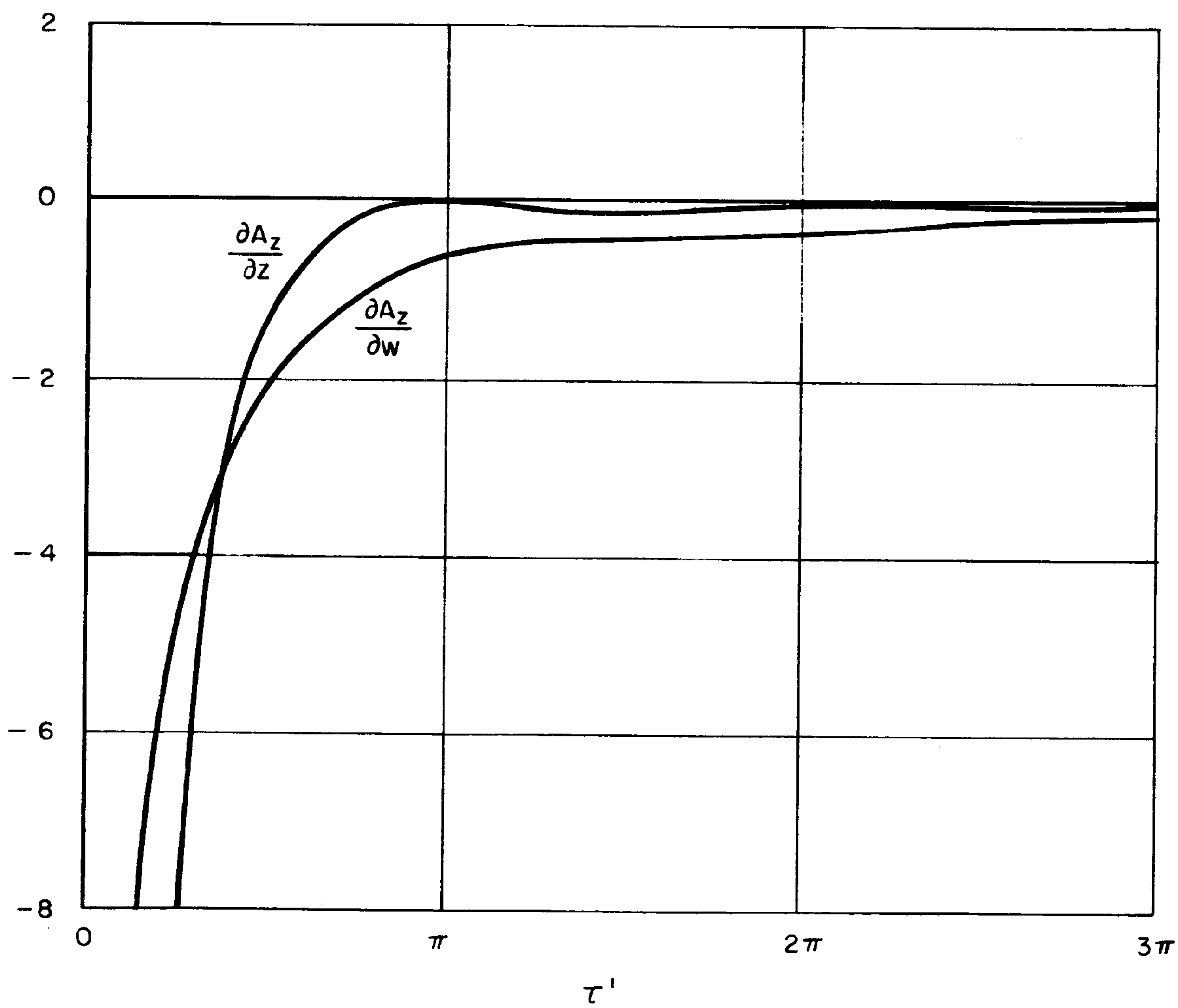
$$A_y = \frac{\partial A_y}{\partial y}\, y + \frac{\partial A_y}{\partial u}\, u + \frac{\partial A_y}{\partial v}\, v$$



GUIDANCE COEFFICIENTS FOR OPTIMUM CONTROL

ORBIT TRANSFER

$$A_z = \frac{\partial A_z}{\partial w} w + \frac{\partial A_z}{\partial z} z$$



GUIDANCE COEFFICIENTS FOR OPTIMUM CONTROL

ORBIT TRANSFER

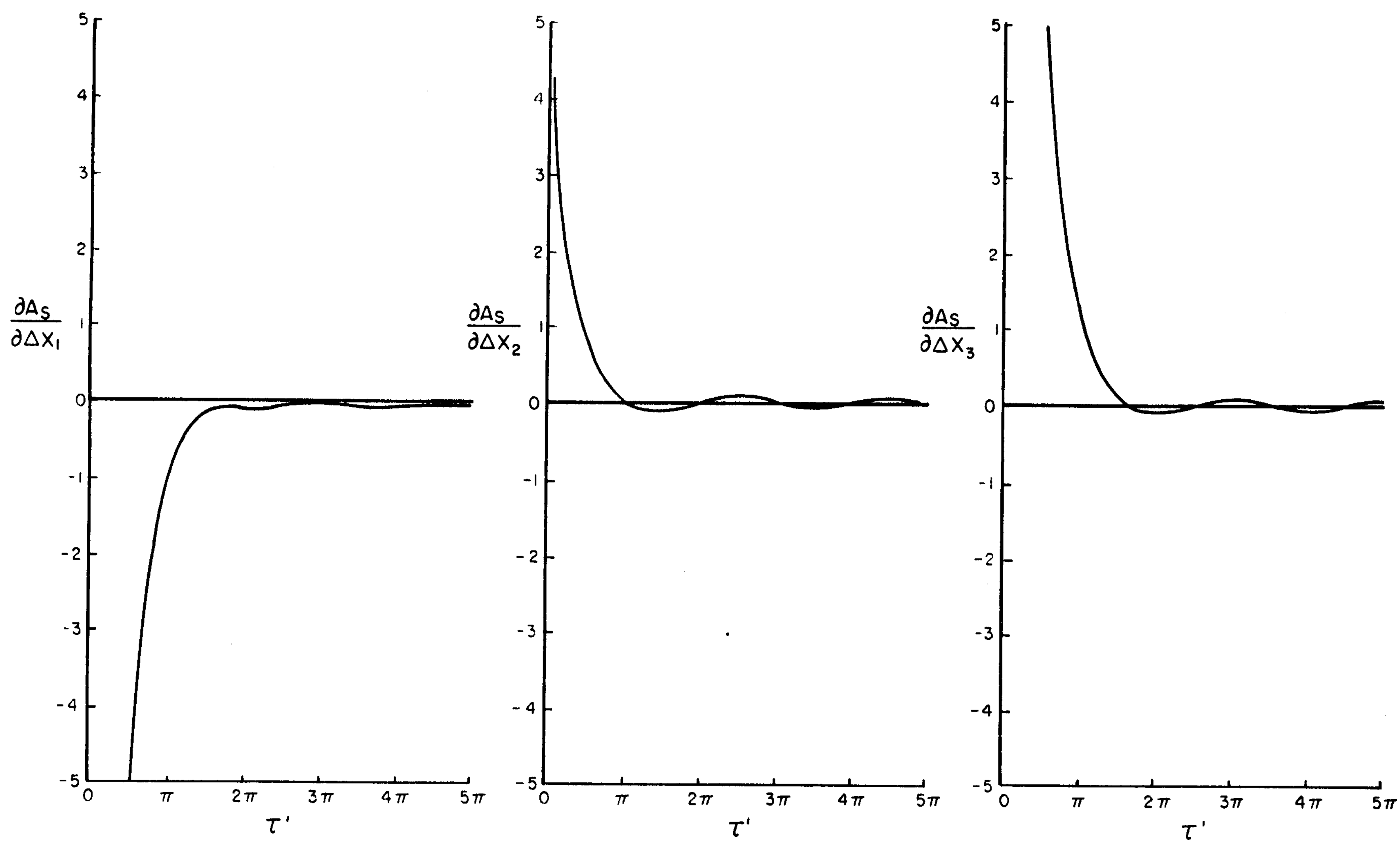


FIG.17

GUIDANCE COEFFICIENTS FOR OPTIMUM CONTROL ORBIT TRANSFER

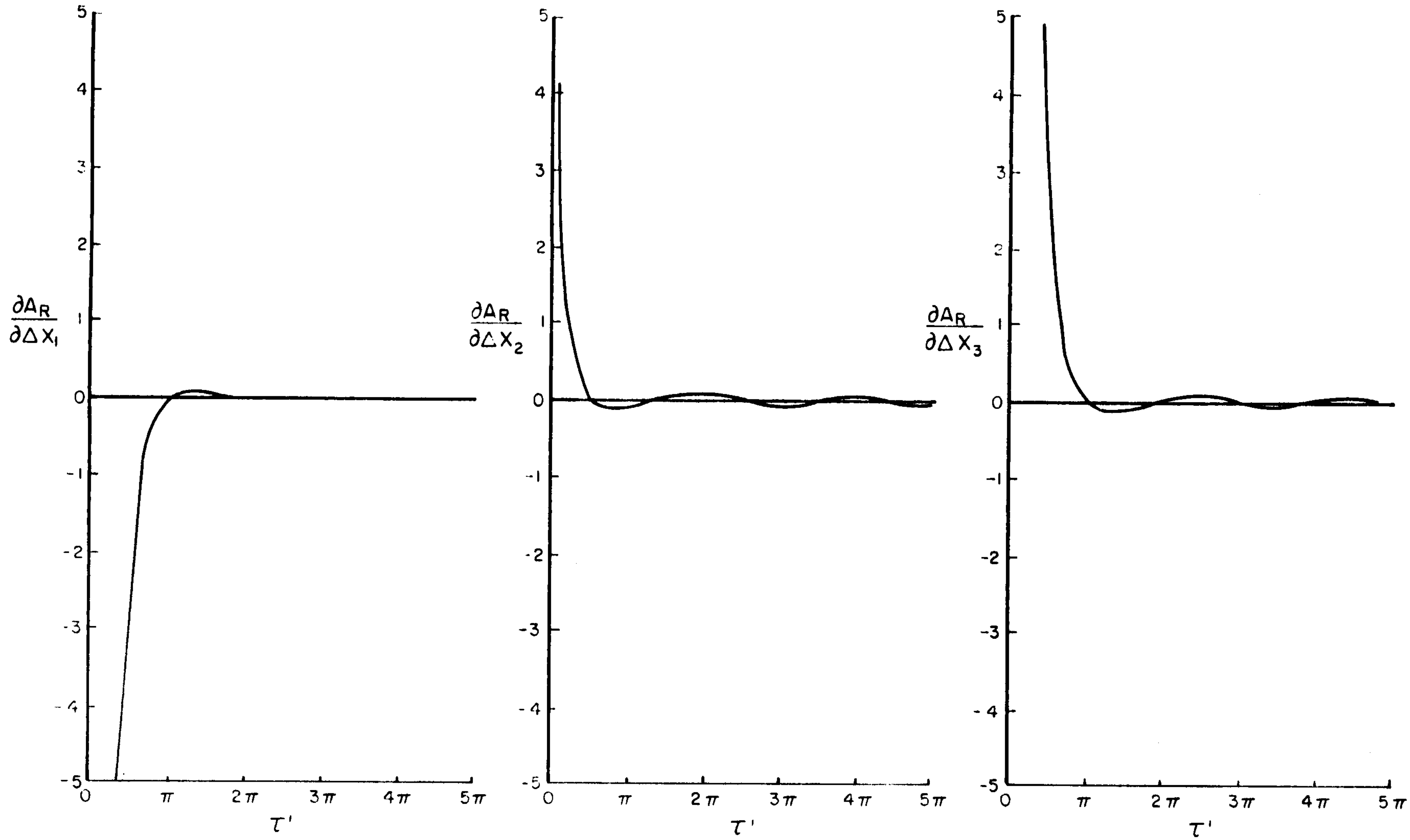


FIG. 18

C-910098-12

FIG. 19

GUIDANCE COEFFICIENTS FOR OPTIMUM CONTROL

ORBIT TRANSFER

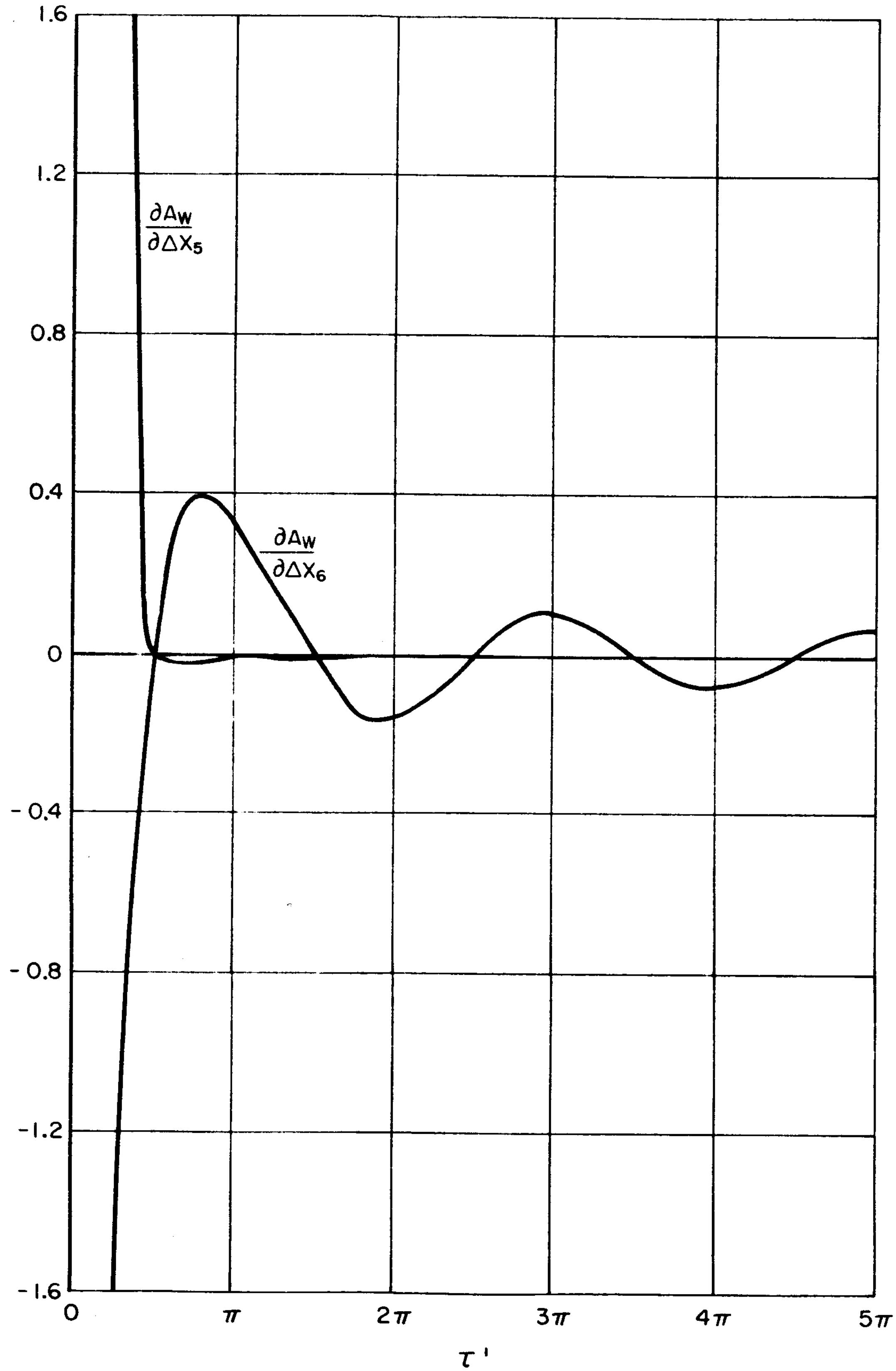
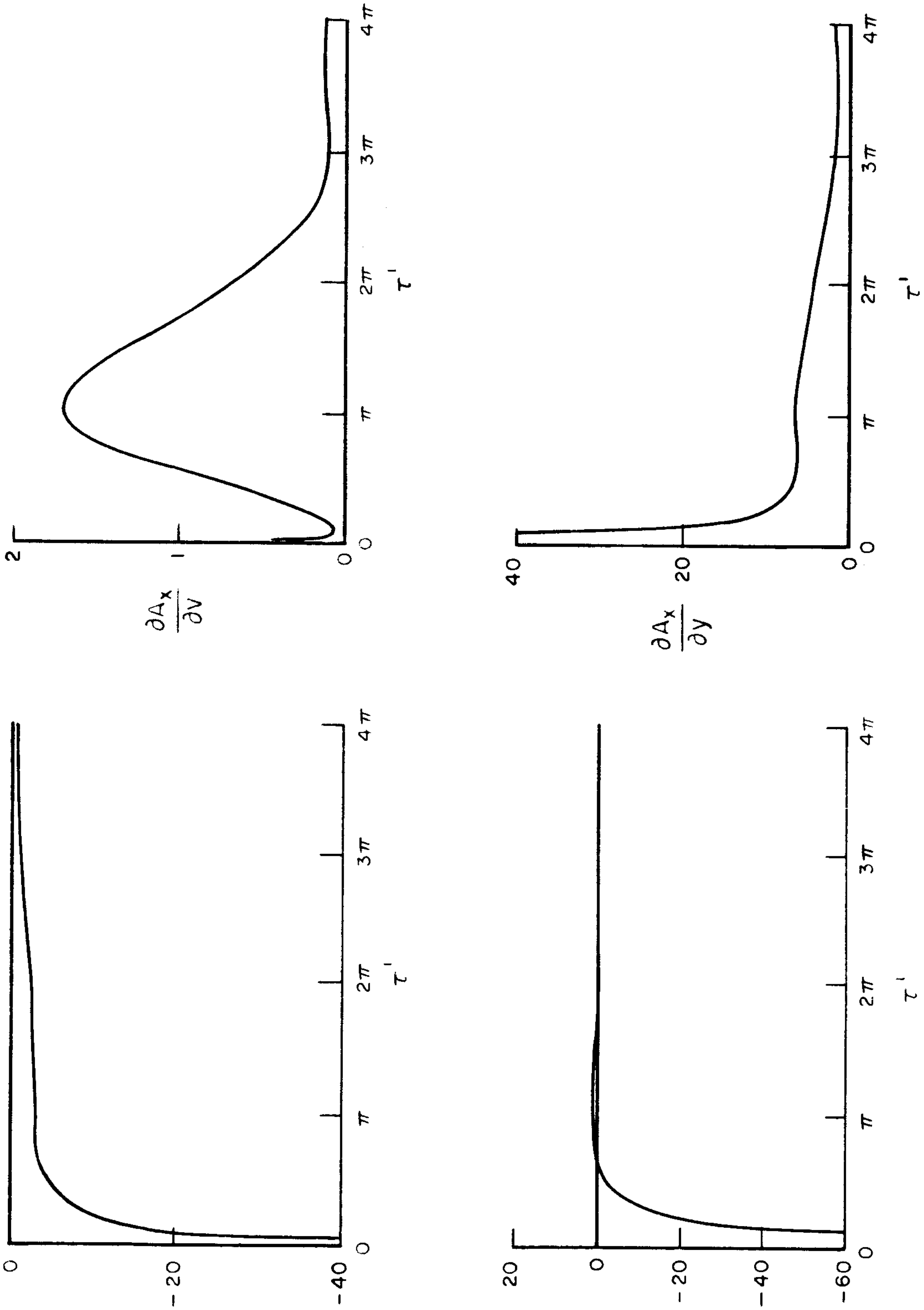


FIG. 20

GUIDANCE COEFFICIENTS FOR OPTIMUM CONTROL
ORBITAL RENDEZVOUS



GUIDANCE COEFFICIENTS FOR OPTIMUM CONTROL
ORBITAL RENDEZVOUS

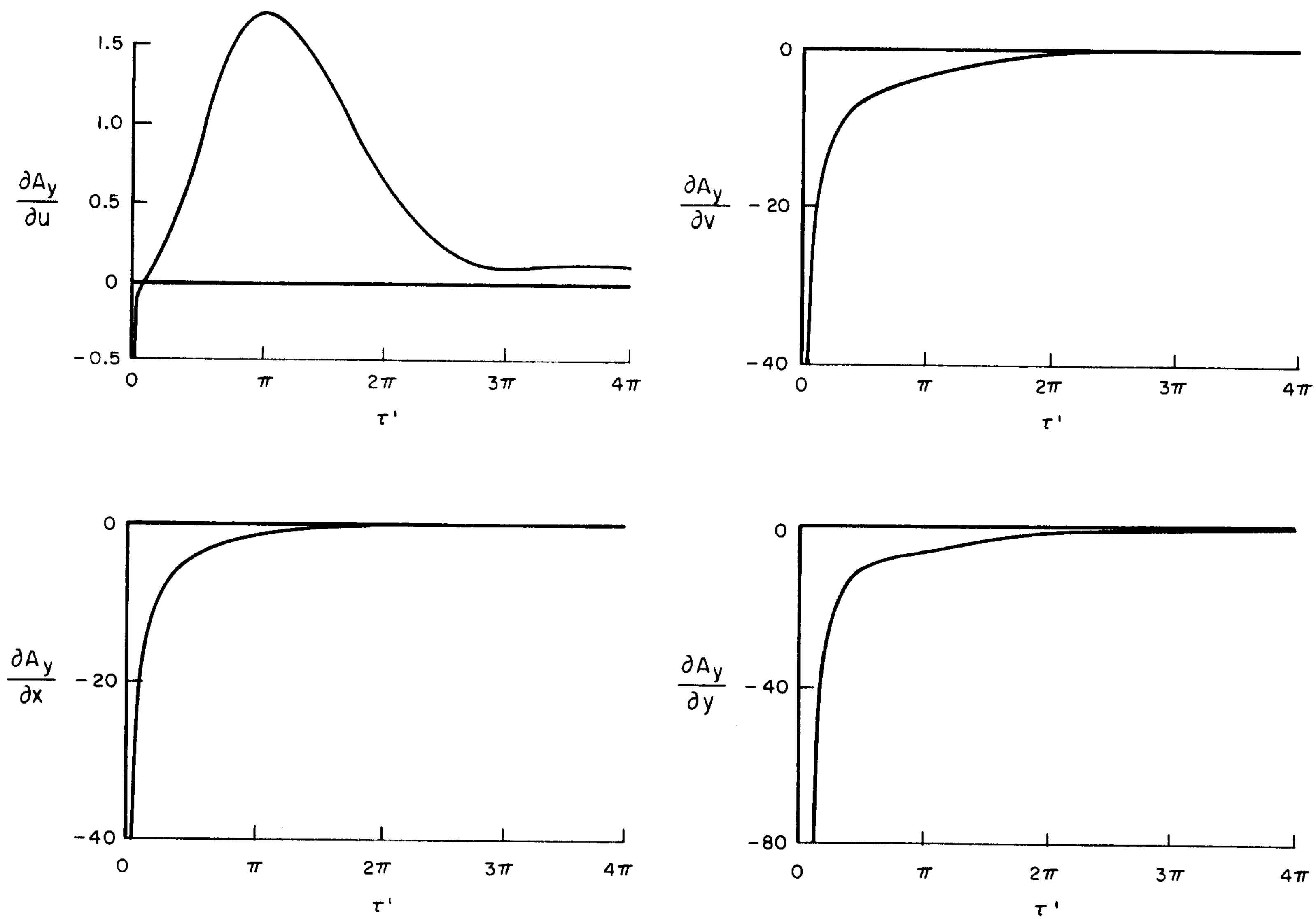
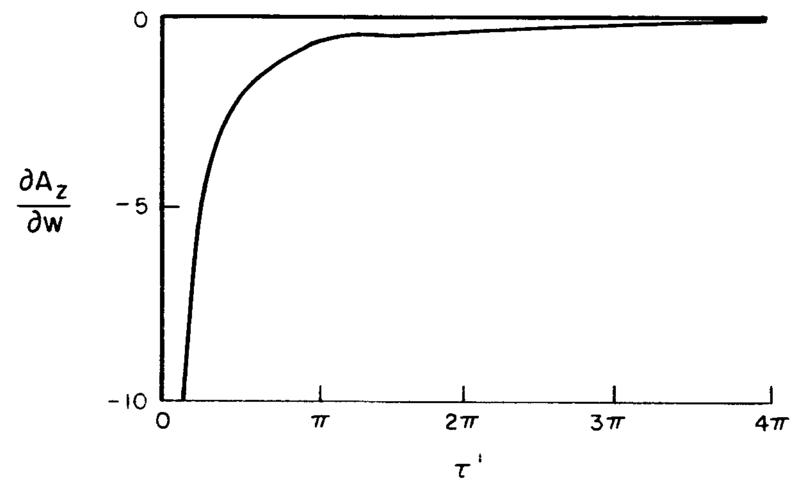
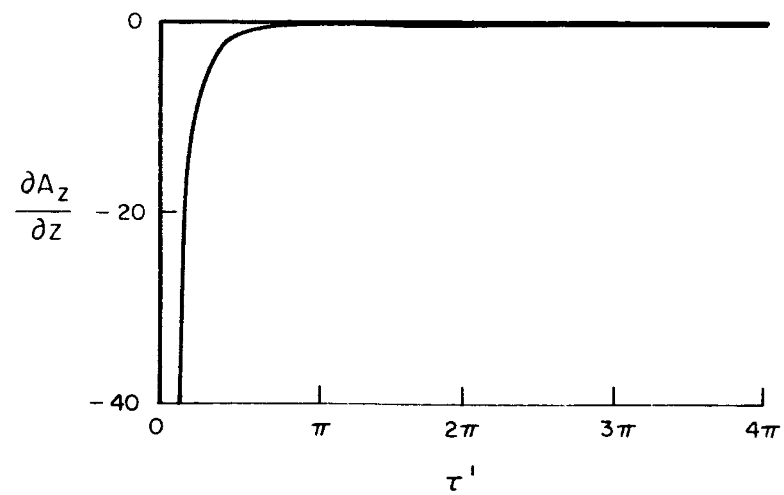


FIG. 21

GUIDANCE COEFFICIENTS FOR OPTIMUM CONTROL

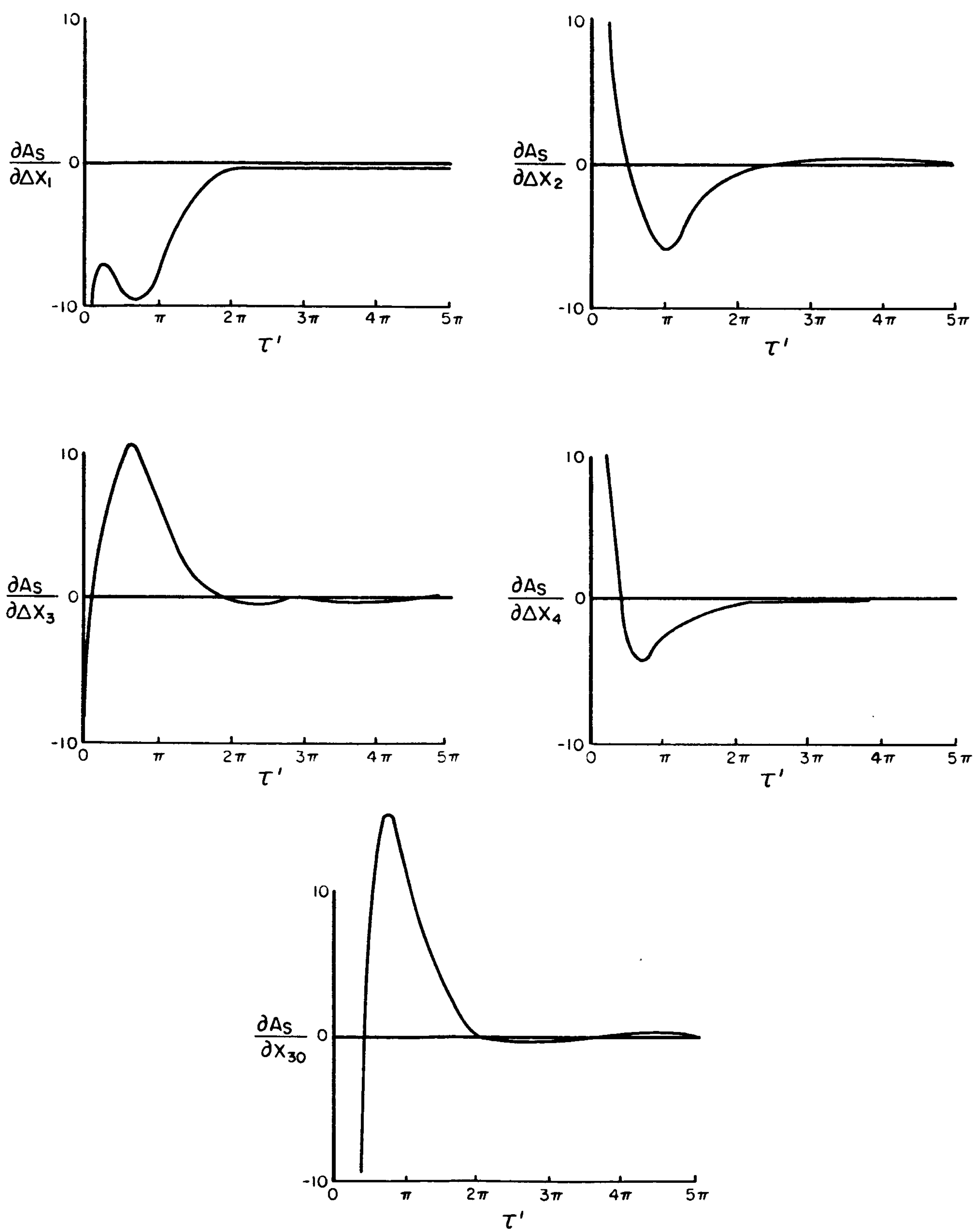
ORBITAL RENDEZVOUS





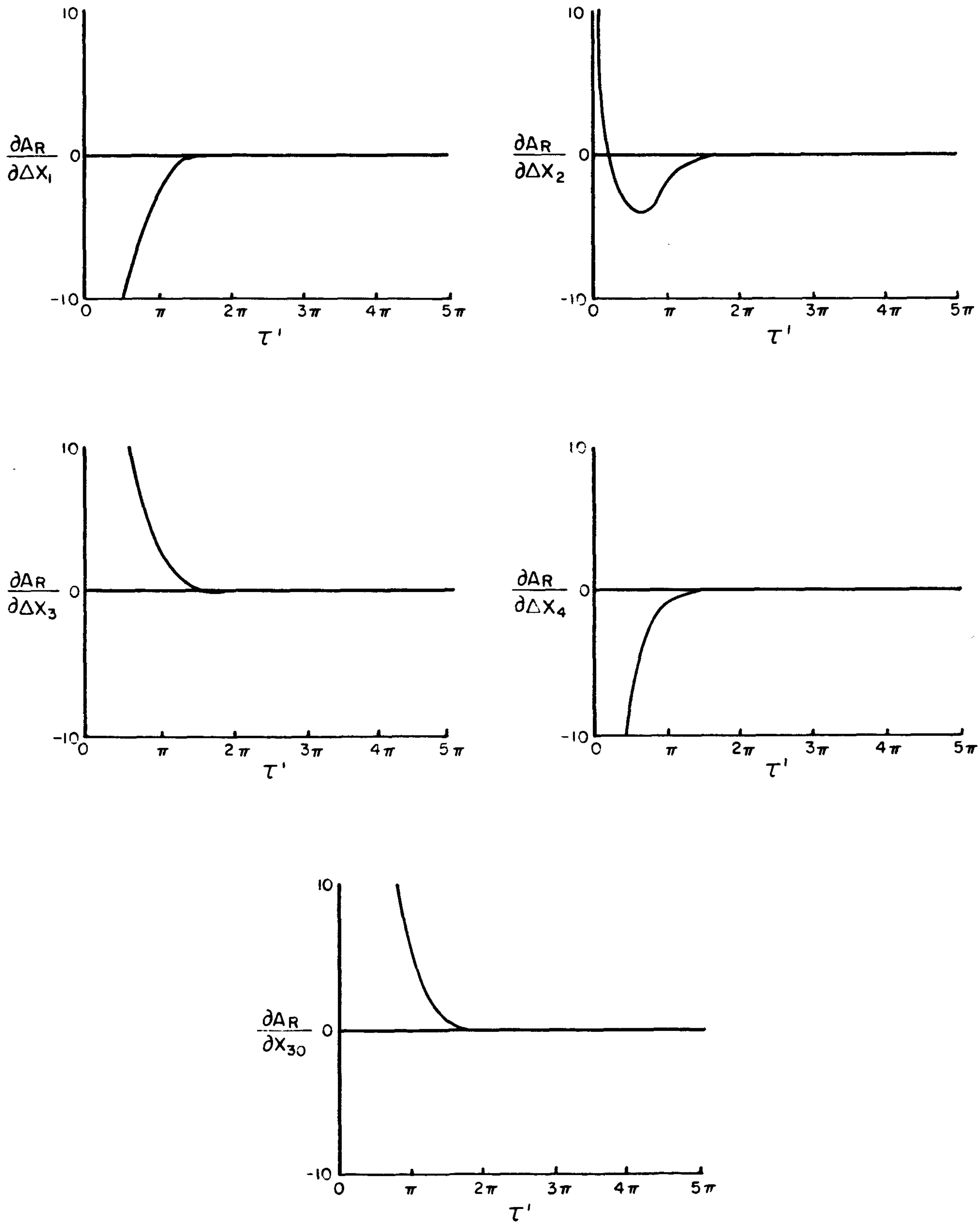
GUIDANCE COEFFICIENTS FOR OPTIMUM CONTROL

ORBITAL RENDEZVOUS



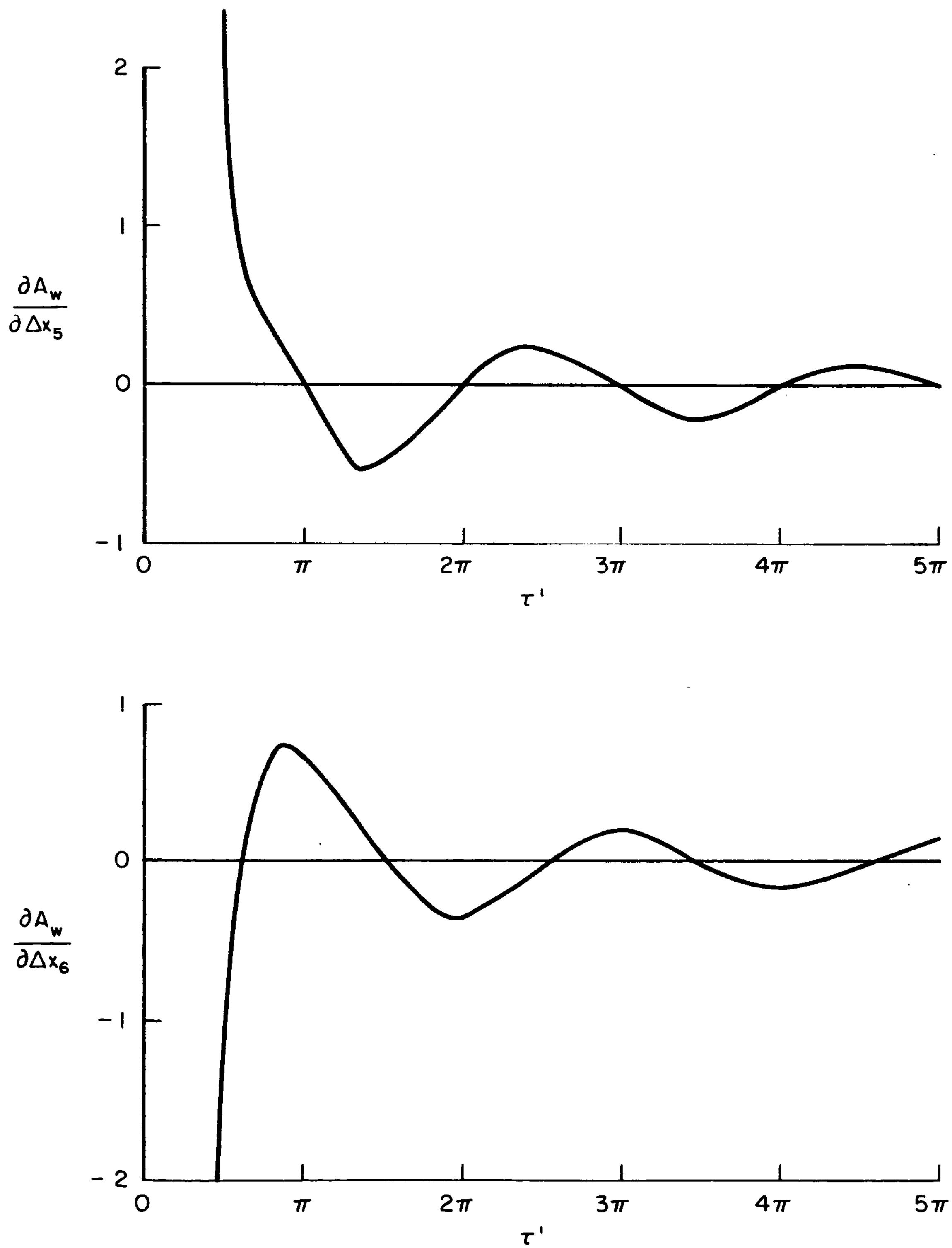
GUIDANCE COEFFICIENTS FOR OPTIMUM CONTROL

ORBITAL RENDEZVOUS



GUIDANCE COEFFICIENTS FOR OPTIMUM CONTROL

ORBITAL RENDEZVOUS



RELATIONSHIP BETWEEN ROTATING AND NON-ROTATING COORDINATE SYSTEMS